THE HISTORY OF ARITHMETIC

KARPINSKI

THE HISTORY OF ARITHMETIC

RECKONING WITH THE PEN RECKONING WITH COUNTERS

The old and the new systems of computation depicted in an early encyclopedic work, entitled *Margarita Philosophica* by Gregorius Reisch, published first in 1503.

THE HISTORY OF ARITHMETIC

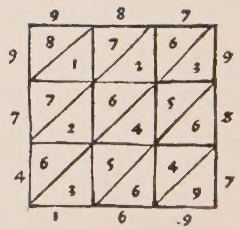
By

LOUIS CHARLES KARPINSKI

Professor of Mathematics, University of Michigan

RAND McNALLY & COMPANY

CHICAGO · NEW YORK



Made in U. S. A

A-25

THE PREFACE

The purpose of this book is to present the development of arithmetic as a vital and an integral part of the history of civilization. Particular attention is paid to the material of arithmetic which continues to be taught in our elementary schools and to the historical phases of that work with which the teacher of arithmetic should be familiar. Particular attention is given, also, to the early American textbooks of arithmetic, printed before 1800, and to the popular treatises on the subject used in England which were the direct source of the American arithmetic.

To understand the progress of arithmetic in America is to understand more fully the whole history of the New World. In this progress the arithmetic of England most directly influenced American arithmetic; but the science of Germany and Italy and Spain and France, the science of the Arabs and the Hindus, and the beginnings of the science of the Egyptians and the Babylonians, all had a working part in the development of our modern science and, in particular, of arithmetic.

The modern tendency in arithmetic is to provide contact with life at as many points as possible. In *The History of Arithmetic* it is shown that arithmetic connects intimately with the early civilization of America and the Orient, that it is associated directly with the progress of the art of printing, and that illuminating contact is made with the development of the English language. All of this contributes to effective teaching in the largest sense.

The study of the history of arithmetic enables the teacher to discriminate between the essential and the nonessential, a particularly important point at the present time when the content of American arithmetic in the schools is changing rapidly.

The author is indebted to a number of librarians and scholars at home and abroad, who have generously contributed information upon many points. To the publishers the writer's thanks are due, as they have not hesitated at the expense involved in the many illustrations, which stimulate interest in the subject and which, by themselves, practically carry the story of the development of arithmetic in its progress from the Old World to the New World.

Louis C. Karpinski

Ann Arbor, Michigan

THE CONTENTS

LIST OF ILLUSTRATIONS

AZTEC HIEROGLYPHICS

Showing the education of the Aztec boy and girl, age 11 to age 14. The food allowance is indicated by the elliptical disks, representing *tortillas* or cornbread.

THE HISTORY OF ARITHMETIC

CHAPTER I

EARLY FORMS OF NUMERALS AND EARLY ARITHMETIC

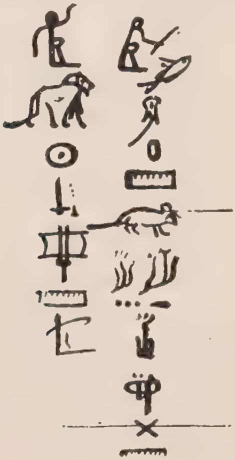
Egyptian Numerals and Arithmetic

Picture writing. Before an alphabet was invented our savage ancestors used pictures instead of words to represent ideas. If a single lion was to be represented, a picture of a lion was drawn; later only the head was drawn. To represent three lions, three of those pictures were made. Early American Indians used this form of writing, known as picture writing, or hieroglyphics. Later some man conceived the bright idea of representing three lions by one lion's head with three strokes under it; five lions by the head with five strokes.

Picture writing is particularly adapted to the representation of numbers. This type of numeral was most highly developed in early Egypt, as much as four thousand years ago.

Egyptians. The Egyptian symbol for 100 may be a surveyor's chain, one hundred units in length. The symbol for 1000 represents the lotus flower, of which there were so many in Egyptian fields. For 10,000 a pointed finger was drawn; and for 100,000 a tadpole was depicted. There were not more tadpoles than lotus

flowers, but probably they seemed like more. For a million the picture of a man with his hands outstretched,



EGYPTIAN HIEROGLYPHICS

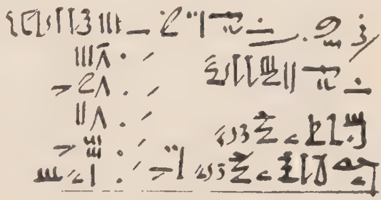
From the Ahamesu papyrus, *c.* 1700 B.C. Represents probably illustration to a problem dealing with grains of corn, sheaf, mouse, cat, old woman. These words, *cat, mouse, sheaf, grain,* represent also the second, third, fourth, and fifth powers of a quantity.

apparently in amazement at so large a number, was used. These symbols are all called *hieroglyphics*, since numbers are represented by pictures of objects.

Decimal system. The Egyptian system proceeds by powers of ten, and so is called a decimal system. This is a natural system to use, since man has ten fingers.

As Egyptian civilization progressed men found the need of some more rapid method of representing numbers and ideas. This advance was made largely by the Egyptian priests, who had time to think about such things. A so-called priestly (hieratic) writing was developed in which shorter ways of writing numerals were used. An Egyptian arithmetical work on papyrus, employing these hieratic numerals, was found in Egypt about seventy years ago. This papyrus is in the British

them as to us. The Egyptians were clever with numbers and with geometrical figures. The pyramids and the obelisks could never have been built as fine as they were without the aid of numbers and geometry. In arithmetic, and, it may be added, in geometry and algebra, the Egyptians made noteworthy progress, establishing the foundations upon which Greek mathematical science rose.



EGYPTIAN PROBLEM

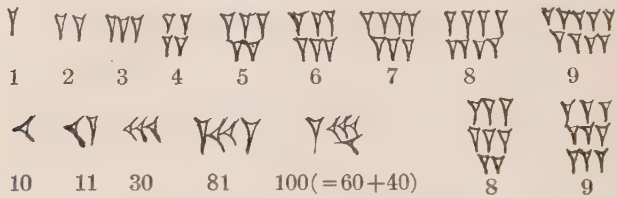
On the distribution of 100 loaves of bread in arithmetical progression among 5 people. Reads from right to left. First line mentions 100 loaves of bread among five people. At the extreme left, in a column, is the arithmetical series, reading down, beginning on second line, 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, 1.

Babylonian Numerals and Arithmetic

Babylonians. The other ancient civilization with which we are most familiar and of which we are reminded several times every day is the Babylonian. Whenever you tell the time of day you pay an unconscious tribute to the ancient Babylonians, for they first divided the day into twenty-four hours and they were the first to divide the hour into sixty minutes. So also the degrees which we use to measure angles, to measure latitude and longitude, all go back to ancient Babylon.

Cuneiform writing. The Babylonians wrote on soft clay with a pointed stick called a *stylus*. Tablets to be kept were baked after the writing was placed upon them. The wedge-shaped characters made in the clay constitute what is called *cuneiform* writing. One hundred years ago nobody could read either Egyptian hieroglyphics or

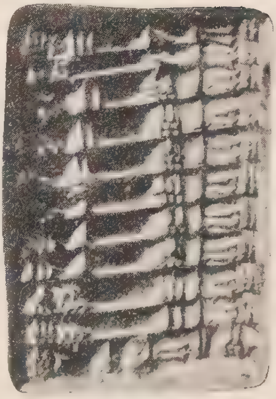
Babylonian cuneiform writing. Today, however, there are many scholars who know how to read these languages that long might properly have been termed "dead." The story of the unraveling of these mysteries records one of the great triumphs of the human intellect.



BABYLONIAN CUNEIFORM NUMERALS

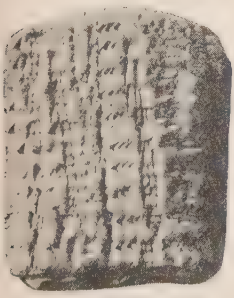
The sexagesimal system — our minutes and seconds. Probably for a long time the Babylonian system of numerals did not go beyond 60. At Senkereh on the Euphrates some old clay tablets were found upon which a Babylonian had written the squares of numbers up to 30. The tablets read easily up to 7^2 is 49. Then the tablet gives for the square of eight: 1–4; since we know that 8^2 is 64, the 1 must stand for 60. The same system was followed throughout these tablets of squares and cubes which furnish a check. Following 8^2, 9^2 is given as 1.21, in the cuneiform characters; the first unit again must represent 60, just as with us a unit in the second place represents 10. Since that time many old Babylonian documents have been found containing these numerals. From the use of the sixty or sexagesimal system we get our minutes and seconds, both in the measurement of time and of angles. The Babylonians were the earliest scientific astronomers, and it was through

astronomy that the degrees and minutes were transmitted to Greece and thus to all of Europe.



BABYLONIAN TABLET
From Temple Library at Nippur, *c.* 1350 B.C. Multiplication table of 18×1, 18×2, etc. In the central column 2, 3, 4, 5, down to 11 in the lowest line; at the right, in column form, 18, 36, 54, 72 (60, 10, 2), . . .

Babylonian multiplication. We know less about Babylonian arithmetic than about Egyptian because of the early Egyptian textbook of arithmetic, written on papyrus, found in an Egyptian tomb, while no similar treatise has been found among the many clay tablets deciphered. However, parts of an old Babylonian multiplication table have been found and many hundreds of tablets going back as far as 3000 B.C. to 3500 B.C. which contain numerals. The Babylonian multiplication table, since the system is a 60 system, extended up to 59 times 59. However, the tables did involve some simplifications. The table of 18 begins 18×1, 18×2, and so on to 18×19, 18×20; then the tables give 18×30, 18×40, and 18×50. Evidently 18×58 would have been obtained as 18×50 added to 18×8. Even with these simplifications the table was difficult, and the series of tablets needed by a computer were too heavy and awkward to be carried about in the pocket.



BABYLONIAN TABLET
Table of squares, *c.* 2200 B.C., from Nippur. Central column gives 30, 31, 32, . . . to 39. At the left are the squares 15 for 900 (15×60); 961 in second line (16·1 or 16×60+1).

Babylonian curvilinear numerals. A second place system of Babylonian numerals was devised, using the blunt circular end of the stylus; in this the crescent was used for a unit with the complete circle for 10. These curvilinear numeral forms were used more than five thousand years ago in the same documents with cuneiform characters, somewhat as we use Roman and Hindu-Arabic

CURVILINEAR NUMERALS

Sumerian clay tablet, *c.* 2500 B.C., in the Harvard Semitic Museum. In the center the number 6; below this 24 indicated by two circles and four half circles.

numerals. The cuneiform type of numerals was always used for the number of the year, for the age of an animal,

BABYLONIAN CURVILINEAR NUMERALS

and in stating that a second or third payment has been made; it was regularly used for the number of animals in accounts concerning the allotment of food. In tablets

giving wages it appears that those actually paid were written in curvilinear and wages due in cuneiform.

The cuneiform characters are also found placed horizontally. Furthermore, on some ancient tablets a system of representation of 100 appears, and on other tablets separate symbols for 600, 3600, 216,000. It must be remembered that what we have somewhat loosely designated as Babylonian civilization covers a period of more than four thousand years and includes at least three historically distinct civilizations.

Babylonian interests. The Babylonians were the most careful bookkeepers of antiquity. The detailed records of ordinary things bought and sold, together with the wages of laborers, including men, women, and children, give us a somewhat comprehensive idea of economic conditions in ancient Babylon. The Babylonians were interested in the mysticism of numbers and in astrology. These interests stimulated them to study both arithmetic and astronomy, so that their priests were able to teach science as well as mysticism to the Greek students who came to them.

Greek Numerals

Archaic Greek numerals. The earliest Greek numerals do not come from Greece proper but are found in excavations on the island of Crete. These antedate by five or six hundred years the Golden Age of Greece.

MINOAN OR CRETAN NUMERAL FORMS OF HIEROGLYPHIC TYPE

The illustration follows the forms given in the work by Sir Arthur Evans, *The Palace of Minos* (London, 1921), p. 279.

Initial letter numerals. In the time of Thales (624–547 B.C.), the first Greek mathematician known to us by name, and for centuries thereafter, the initial letters of the words for five, ten, one hundred, one thousand, and ten thousand were used to represent the corresponding numerals.

M	X	H	Δ	Π or Γ
10,000	1000	100	10	5
μύρια	*χίλια*	*ἑκατόν*	*δέκα*	*πέντε*

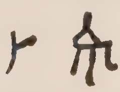
English words:

myriad	*kilometer* *kilogram*	*hectare*	*decimal*	*pentagon*

Four straight lines were used to represent 4; four symbols Δ for 40. The 5 symbol was combined with the higher symbols to give 50,  or  500, 5000, and 50,000.

50 700 8000 50,000

Alphabet numerals. About five hundred years before the Christian Era, 500 B.C., a new and more compact system of number symbols was introduced into Greece. The first nine letters of the Greek alphabet were used for 1 to 9; the next nine letters were used to represent 10, 20, etc., to 90; the final group of nine letters was used for 100, 200, 300, etc., to 900. Three older letters not found in the present Greek alphabet were introduced to make



ANCIENT SYMBOL For "Drachmas 1000" in Elephantine papyrus, *c.* 300 B.C.

the necessary 27 letters. By placing a stroke before a letter it multiplied the number represented by 1000; thus ͵β represented not 2, but 2000. The older myriad symbol (for 10,000) was frequently used with this system; thus $\overset{\beta}{M}$ for 20,000.

α	β	γ	δ	ε	ϝ	ζ	η	θ
1	2	3	4	5	6	7	8	9
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
10	20	30	40	50	60	70	80	90
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
100	200	300	400	500	600	700	800	900

Hebrew letter numerals. The Hebrews used this same system with Hebrew letters, and the Arabs continued its use up to 800 or 900 A.D. This system is more compact than is the initial letter (or Attic) system. But the multiplication table is much longer than with our numerals. Thus

$2\times3=6$ $\quad$ $2\times30=60$ $\quad$ $20\times3=60$ $\quad$ $20\times30=600$

would be

$\bar{\beta}$ τῶν $\bar{\gamma}$ $\bar{ϝ}$ $\quad$ $\bar{\beta}$ τῶν $\bar{\lambda}$ $\bar{\xi}$ $\quad$ $\bar{\kappa}$ τῶν $\bar{\gamma}$ $\bar{\xi}$ $\quad$ $\bar{\kappa}$ τῶν $\bar{\lambda}$ $\bar{\chi}$

The numerical connection of these products is not evident in the letter products, making each one a separate thing to remember.

Commonly when letters were written with numerical value a bar was placed over the letters to show that a number and not a word was intended. Occasionally a play on this system was used by giving instead of a name the number made by the letters. In the Bible in Revelation "the number of the beast" is given as 666, or more correctly 616; this refers probably to "Nero Caesar," spelled in Hebrew letters which can be made to total

666 (616). Thus the letters $\kappa\alpha\tau$ would have the number $20+1+300$, or 321 would be the number of "kat."

Greek tablets. The Greeks frequently wrote on a wax tablet with a sharp-pointed stick, the stylus. The ancient tablet resembles an old slate. Upon such a tablet a Greek child would write his letters; at least one tablet has been found on which probably a child wrote the multiplication table, beginning $\alpha\ \alpha\ \alpha$, $\alpha\ \beta\ \beta$, ... Sometimes Greek geometers used a board covered with sand, in which figures were easily drawn and easily erased by smoothing out the sand. The great Archimedes is reported to have been engaged in drawing a diagram on the sand when he was killed by a soldier of Marcellus, after the fall of Syracuse.

Arithmetic and logistic. Concerning the operations of addition, subtraction, multiplication, and division with Greek numerals no treatise has come down to us. The Greeks divided the subject of arithmetic into two parts. The one subdivision called *arithmetica* was purely theoretical arithmetic corresponding to our modern number theory; the other, called *logistica*, was devoted to computation.

Greek practical arithmetic. An ancient scholion (commentary) on a work of Plato informs us that *logistica* "is useful in the relations of life and business"; also that it treats "the methods called Greek and Egyptian for multiplication and division, as well as the summation and decomposition of fractions." Similar information is given by Proclus in the fifth century A.D. The indication of continued Egyptian influence is seen both in the

multiplication and in the reference to fractions. The Greek letter numerals made computation difficult, which may explain the fact that the Greeks had no fondness for computing. Greek children had some drill in the multiplication table and in addition with their numerals, and also undoubtedly drill upon representing numbers with the fingers and upon an abacus with little stones.

Speculative arithmetic. The speculative arithmetic of the Greeks engaged the attention of students over such a long period of time that it is worthy of attention as interesting from the pedagogical point of view. The terminology of present-day arithmetic and some current phrases bear evidence of the continued influence of the mystic element in numbers. Such expressions as "luck in odd numbers," "lucky seven," "come seven, come eleven," and "all good things are three" carry us back in spirit and even in content to the mysticism of numbers as practiced first in Babylon and then in Greece and Rome.

Among the Greeks two divergent methods of treating the same arithmetical facts were followed even from the time of Pythagoras. On the one hand there was the strictly mathematical treatment which is exemplified by the arithmetic of Euclid (*c.* 320 B.C.), appearing in Books VII, VIII, IX, and X of the *Elements;* on the other hand there was the simple statement of arithmetical facts, without any proof but with philosophical disquisition, as exemplified by the arithmetic of Nicomachus (*c.* 100 A.D.).

Odd and even. Odd and even numbers are two of the great subdivisions of numbers, probably first so set apart in Egypt; among the Greeks even numbers were further subdivided into two or three subdivisions. Questions

concerning the divisibility of numbers suggested the classification of numbers into prime and composite, and suggested problems like that of the greatest common divisor and least common multiple, all of which are treated with logical rigor by Euclid.

Perfect, superabundant, and deficient numbers. A perfect number was defined by the Pythagoreans as one which is equal to the sum of its divisors. Consider the two series,

	1	2	4	8	16	32	64
and	1	3	7	15	31	63	127,

in which each lower number represents the sum of the upper series of numbers up to and including the number immediately above itself. Euclid proves that whenever the lower number is a prime number the product of upper and lower numbers is a perfect number. Thus 6 is perfect since the divisors of 6 are 1, 2, and 3, whose sum is 6; similarly 28 (4×7) and 496 (31×16) and 8128 (64×127) are perfect, since 7, 31, and 127 are prime numbers.

Euclid contents himself with perfect numbers, whereas in the Greek treatises on *arithmetica* by Nicomachus of Gerasa and Theon of Smyrna, both writing probably in the second century of the Christian Era, superperfect and deficient numbers are also discussed. Theon contents himself with numerical illustrations, whereas Nicomachus rambles on as follows:

"(1) But there appears as a mean between these two kinds already considered, that is as it were opposed in the manner of extremes, the so-called perfect number which is found in the realm of equality. This is a number that neither makes the sum of its own parts greater than itself nor shows itself greater than the sum of its parts, but is always equal to the sum of its parts. Now that which is equal is

always regarded as midway between the more and the less and is, so to speak, moderation between the excessive and the deficient, the harmonizing tone between that which is too high and that which is too low. (2) Whenever, then, a number neither exceeds in amount all its parts, after all that it may contain have been combined and added up and compared with itself, nor is surpassed by them in amount, then such a number is properly called perfect, since it is the number that is equal to the sum of its own parts. For example, the numbers 6 and 28; for 6 can be divided into one-half, one-third, one-sixth, which are 3, 2, 1, and these added together make 6, which is equal to the original number, being neither more nor less. And so 28 has as its parts a half, a fourth, a seventh, a fourteenth, a twenty-eighth, which are 14, 7, 4, 2, 1, and these added together make 28. And so neither are the parts more than the whole, nor the whole greater than the parts, but the comparison results in equality, which is the peculiar character of the perfect. (3) And it is a fact that just as the things that are beautiful are seldom found and easily numbered, while the things that are ugly and base are manifold, so also the numbers that are superabundant and deficient are found to be numerous and irregular in their series and the discovery of them is at haphazard; but the perfect numbers are easily numbered and arranged in fitting order. For there is only one found in the units, *viz.*, 6, and only one other is found in the tens, 28; and the third and only in the hundreds is 496, and the fourth is that in the order of the thousands, that is to say, below 10,000, *sc.* 8128. And it is characteristic of all these numbers that they alternately end in 6 or 8, and are always even numbers."[1]

The above quotation constitutes only one-third of the talk about perfect, superabundant, and deficient numbers indulged in by Nicomachus. The further perfect numbers do not alternately end in 6 and 8; the next one is 33,550,336 and not in the ten thousands, as implied.

[1] This quotation is taken from the work, "Nicomachus of Gerasa, Introduction to Arithmetic," by M. L. D'Ooge, with studies in Greek arithmetic by F. E. Robbins and L. C. Karpinski, *University of Michigan Studies*, Humanistic Series. (In press.)

Number series. The three series

1	2	3	4	5	6	7	8	9	10	....
1	3	5	7	9	11	13	15	17	19	....
1	2	4	8	16	32	64	128	256	512	....

play a great rôle in Greek arithmetic. Thus the fact that the sum of the sequence of odd numbers beginning with 1 always gives a square number is noted and is considered a theorem of beauty, as indeed it is. Nicomachus notes that the following sums are cubes:

1	$3+5$	$7+9+11$	$13+15+17+19$
1^3	$8=2^3$	$27=3^3$	$64=4^3$

$$21+23+25+27+29$$
$$125=5^3$$

and the next six odd numbers give 216, or 6^3, and so on forever.

Figurate numbers. One other peculiar interest of the Greek *arithmetica* is reflected in our terms "square" and "cube" as applied to numbers. The Greeks considered numbers as generated by points, and so classified numbers as linear (prime), and as plane and solid, with respectively two and three factors. A triangular number is one which can be built up as the sum of the sequence

1 2 3 4 5 ....;

geometrically

1 3 6 10 15 ....

Similarly, the square numbers are built up as sums of terms beginning with 1, in the series

1 3 5 7 9 ....

or geometrically

Thus the Greeks went on with pentagonal numbers, built up from the arithmetical series,

1 4 7 10 13 ....

In their enthusiasm for "figurate" numbers some misguided arithmeticians spoke of the number of a horse or dog from a geometrical drawing of the animal by dots.

Probably our word "figures" as referring to numbers, and hence "figuring," can be traced to this Greek interest communicated to Latin Europe by Boethius and for a thousand years taught as a part of arithmetic in the church schools of Europe.

Roman Numerals and Roman Arithmetic

Roman numerals. The Roman numerals in the forms which we employ would appear to be alphabetic in character. However, with these symbols as with others, only the most ancient forms can give us any information concerning the true origin. The earliest forms for 50, 100, and 1000 were apparently derived from the Chalcidian alphabet, but there is no indication that they are initial letters of the corresponding words. The ten symbol is supposed to have been derived from some form of crossing

out nine units, ; the symbol for twenty, , which is found confirms this view. The symbols for five, five hundred, five thousand, and fifty thousand were obtained by halving the symbols for ten, one thousand, ten thousand, and one hundred thousand, respectively.

The C for one hundred is found on early monuments; possibly the transition to this letter from some earlier

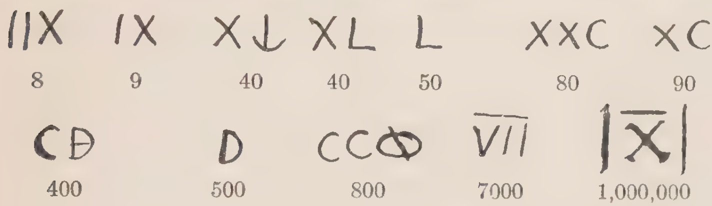
ORDINARY ROMAN NUMERAL FORMS

The other tens and hundreds are built up in precisely the same manner by addition, as indicated above for 60, 90, 400, 900.

form was made because *centum* begins with the letter. However, in the early Latin the initial letter of *centum* was rather "K." The student does well to remember the connection between *centum* and such words as "cent," "century," and "centimeter."

The ancient numeral for one thousand in Latin inscriptions resembles a Greek Θ placed horizontally. M is used occasionally as symbol for *milia* in the expression *milia passuum*, from which the word "mile" is derived. M is

not used in early inscriptions as a numeral in combination with the other symbols. The symbol for fifty in the early



ROMAN NUMERAL FORMS

The forms in the above two lines are rather less commonly used than those on page 20.

These forms and those on page 20 have been copied directly from photographs of Roman monumental inscriptions.

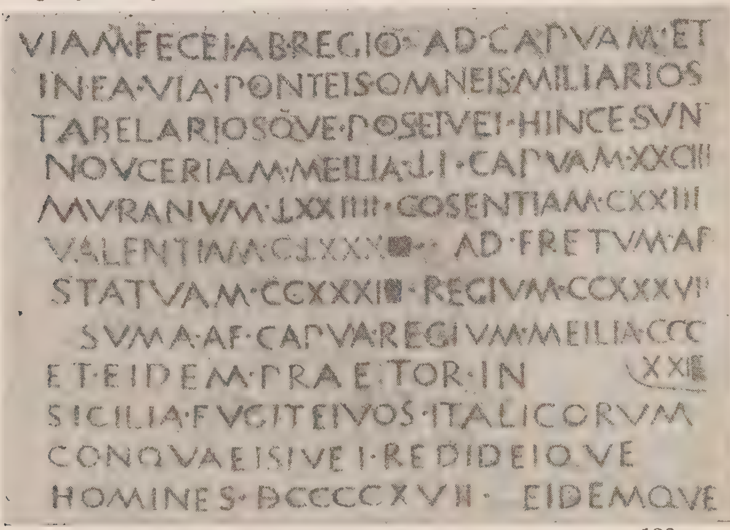
forms is only rarely an L; similarly, the symbol for five hundred is not a D but rather one-half the symbol for one thousand.

30,000 SESTERTII AS FOUND ON AN EARLY ROMAN MONUMENT

Variant Roman forms. Thousands were occasionally indicated by a bar above a given numeral; $\overline{\text{XVIII}}$ for eighteen thousand. From the time of Hadrian inscriptions are found which indicate thousands by a bar above and vertical bars at the side, but in general this notation was used for hundred thousands. Thus $|\overline{\text{X}}|$ for one million or ten times one hundred thousand.

The method of writing millions and other large numbers in the time of Caesar and Cicero varied; in general, writers employed the words in full, in the form of "thouands of thousands" and the like.

Subtraction in symbols. The subtractive principle was employed by the Romans as a convenience when space



MILESTONE AND SIGNBOARD GIVING DISTANCES FROM RHEGIUM, 130 B.C.

In the fourth line is our word "miles" in the ancient spelling *meilia*, followed by 51. The numeral at the end of this line is 83, written in the subtractive form because there was not room for LXXXIII. In the fifth line in the numerals for 74 and in the numeral forms which follow the subtractive principle is not employed.

limitations necessitated abbreviations. The full forms, like XXXX for 40, are the common Roman forms. It is worth noting that occasionally the Babylonians used the subtractive principle in writing numbers like 18 and 19. The Latin words for 18 and 19 suggest the forms, reading *duo de viginti* and *unus de viginti* or "two from twenty" and "one from twenty."

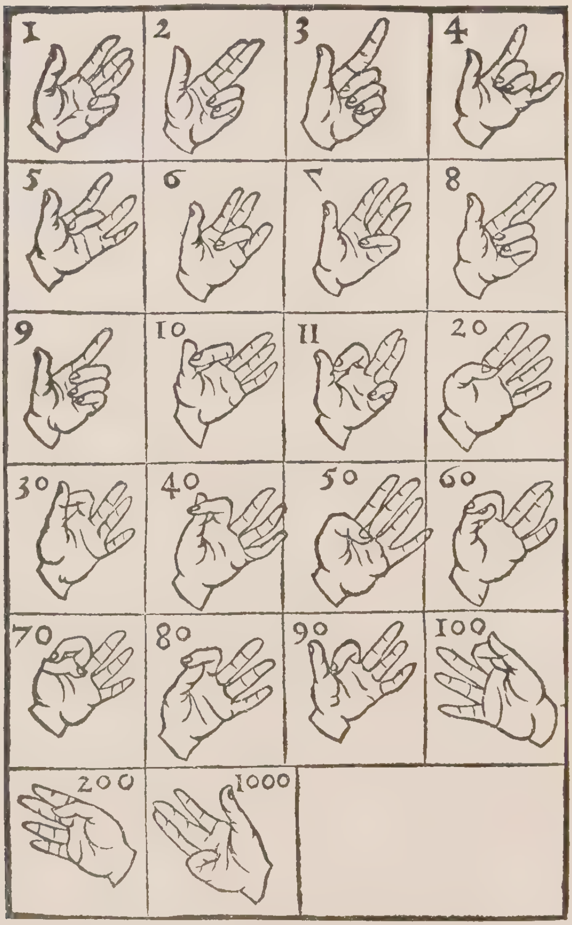
Roman arithmetic. Concerning Latin instruction in arithmetic in classical times we have nothing more definite than concerning the Greek. Again the use of fingers and

abacus were quite certainly taught to children. However, this instruction was not considered a fundamental part of the education, and contemporary discussion of it is only accidental and incidental.

Roman fractions. The Roman fractions, as we shall note later, did leave an impress, albeit an unfortunate one, upon our system of weights and measures. Aside from this the civilization of Rome exerted only indirect influence upon mathematical science. The technical vocabulary of mathematics (*see* Chapter VII) largely traces back to Latin, but primarily because Latin continued so long the language of the schools. The early commercial arithmetic which reached its highest development in Italy certainly owes some of its practices and its attention to detail to the legal genius of the Romans which so profoundly affected European institutions.

The Fingers and the Abacus

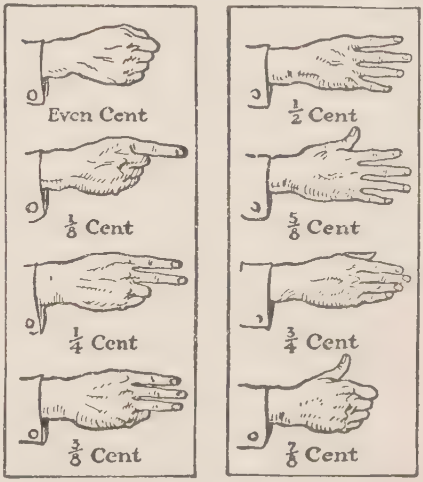
Finger reckoning. An entirely different system of representing numbers is by use of the fingers (Latin, *digiti*). The Greeks used this type of representation and it is still used by savage races of Africa, by Arabs, and by Persians. In North and South America the native Indian and Esquimo tribes use the fingers, and many of their words for numbers refer to fingers and hands, just as the word "digits" traces to the use of fingers. In medieval Latin the word *articuli* was used to indicate pure tens or hundreds, referring to the joints of the fingers employed in the representation of tens. The illustration on page 24 shows the use of the fingers as taught in the early printed books on arithmetic.



FINGER RECKONING

This illustration is from the works of Noviomagus (Bronkhurst), *De Numeris* (Cologne, 1544). Similar illustrations appear in some editions of Recorde's arithmetic and in Paciuolo's great Italian treatise of 1494. The Venerable Bede wrote, probably early in the eighth century A.D., a treatise on the subject, explaining this system. Hundreds on the right hand follow the tens on the left, and thousands are like units on the left.

Two peculiarities are worth noting. Practically all people begin with the little finger of the left hand to represent one unit. Further, the American Indians generally count up to twenty using both fingers and toes, which when you are barefooted is quite easy. Among the Mayas of Yucatan twenty was the base of their number system, used similarly to the way that we use ten. The Esquimos and the Indian tribes along the west coast of North America use twenty generally as the base, but they do not carry the system as far as did the Mayas.

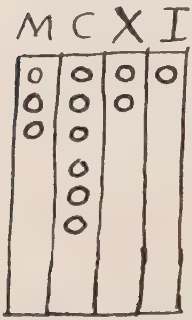


MODERN FINGER RECKONING

The fingers of the left hand are used in this way on the floor of the greatest grain market in the world, the Chicago Board of Trade. Price is indicated always with reference to the last sale, by the hand held with palm toward the buyer and with palm outward for a broker trying to sell. A broker wishing to buy 5000 bushels of wheat at $106\frac{5}{8}$ holds his five fingers outstretched, palm in. Another broker nods acceptance; the buyer indicates 5000 by a single finger held vertically.

Abacus reckoning. For large numbers the fingers were found inconvenient. A system of recording numbers by using small stones on an abacus was used by early Egyptians, Greeks, and Romans, and a variation continues in use today in China, Russia, and Persia. The Chinaman in America who runs a laundry or a store generally uses an abacus in the form of beads on a wire frame.

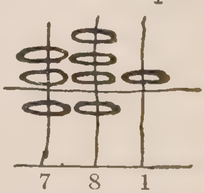
The Romans used on the abacus little stones or *calculi* from which we have the word "calculate"; similarly in Greek the words for "stone" and "to calculate" have a common stem. A stone in the first column represents simply one unit; a stone placed in the second column represents ten; in the third column it represents one hundred; a stone in any column represents ten stones in the next column to the right. In adding two numbers on the abacus when you have ten stones in any one column you take them up and *carry one* stone over to the next column. Or in subtraction you may *borrow one* stone from the next column to the left to make ten in the right-hand column.



ROMAN ABACUS WITH LITTLE STONES

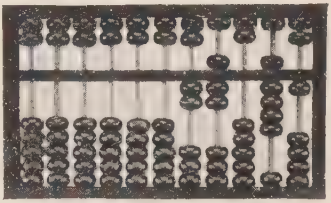
On this abacus is represented the number 3621. This abacus could be made with as many columns as one pleased. Pope Gerbert (Sylvester II, *c.* 1000 A.D.) is said to have had one made with 27 columns. Counters marked with numerals 1 to 9 were sometimes used.

On the Chinese *suan-pan*, meaning "reckoning board," and the Russian abacus, and the Japanese machine for computation are found beads strung on wires. In the Chinese form a rod separates each wire into two parts, one with five beads and one with one bead; the Japanese use five beads and two beads. A single bead from the two represents five units; thus 781 is represented by one bead from the five on the units' wire; one bead from the two and three from the five on the tens' wire; one from the two and two of the five on the hundreds' wire.

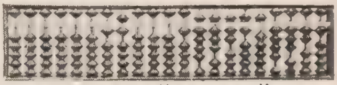


Only the beads employed are shown.

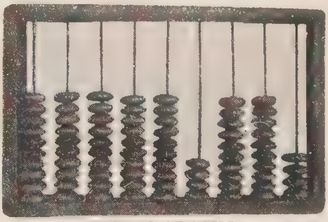
Fundamental operations on the abacus. The earliest treatises discussing the fundamental operations upon the abacus date from the tenth century A.D. The details of ancient usage are not known. Division is the difficult operation; during the Middle Ages a method of division on the board by completing the divisor to one hundred or to a multiple of ten was introduced. This was called "golden division" as opposed to the ordinary, called the "iron division." It is worthy of note that Chinese computers become so expert with the abacus that they can carry through long computations more rapidly than an expert computer can in writing.



CHINESE "SUAN-PAN"
The number 27,091 is represented.



JAPANESE "SOROBAN"
1987654321 is represented at the right; 16 in the two columns to the left of the center.



RUSSIAN ABACUS

Decimal system universal. Herodotus pointed out that the use of ten quite universally as the base of number systems is undoubtedly due to the fact that we have ten fingers. In his day and even to the present day children have used their fingers for computation purposes, with this difference that for centuries among the Greeks and Romans formal instruction was given in the use of the fingers.

Tangible arithmetic. Stones on an abacus were used as we have indicated by the Latins and by the Greeks,

by Chinese and Russians in the form of beads on a wire, by medieval Europe in the form of counters thrown on lines, and such devices continue in wide use today among the Chinese, Persians, and Russians. Certainly more individuals are doing arithmetical sums by machinery today than by the processes of our arithmetic; possibly we might even exclude from the comparison the more recent widely used mechanical computers, which have taken such a burden from the shoulders of individuals condemned to endless computation. Possibly the schools of tomorrow will teach the use of machines to eliminate the work spent on multiplication tables and on long division.

Chinese, Japanese, and Korean Arithmetic

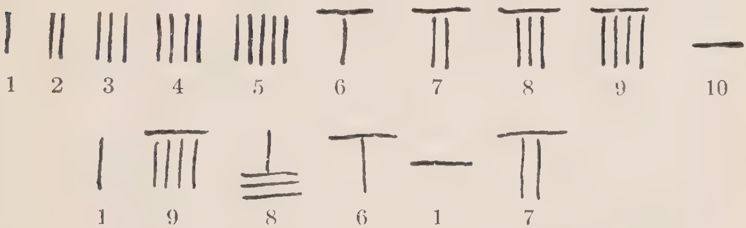
Ancient systems. The Chinese arithmetic lays claim to an antiquity which approximates that of Egypt and Babylon. Although the dates are not established as precisely as might be desired, nevertheless the development of arithmetic centuries before the Christian Era appears to be certain. In large measure the early arithmetic of Korea and Japan was based upon the Chinese.

Tangible arithmetic. The most striking fact concerning the early development of operations with numbers is that tangible methods of representing numbers were employed. Before the abacus or *suan-pan* appeared, the Chinese used little rods, called *sangi*, to represent numbers. These rods were adopted by the Japanese and the Koreans, who continue to employ the computing rods to the present day.

Decimal system. The numerals were represented in powers of ten upon a checkered board, later replaced

by lines. After the zero was introduced zeros were used with the rods to indicate vacant places. In algebraic processes a stroke or rod placed diagonally across a number indicates that the given number is to be subtracted.

The abacus. About the twelfth century the abacus was introduced into China, and that continues in use there until the present day. The Japanese later modified the Chinese *suan-pan*, making a more logically correct



CHINESE AND JAPANESE "SANGI," OR NUMBER RODS

KOREAN NUMBER RODS

Sometimes made of bones and called "Korean bones."

instrument. The Japanese *soroban* is made with a sufficient number of columns so that two numbers can be represented upon the instrument simultaneously, as in multiplication or division; any column can be taken to represent units.

Practical problems. The problems of the early Chinese arithmetic as found in the classical work entitled *Chiu-chang* are similar to those of early India and to those of the Arabs in that mensuration and commerce and alligation and other practical topics receive attention in

addition to abstract problems whose nature is concealed under the oriental phrasing.

"If 5 oxen and 2 sheep cost 10 taels of gold, and 2 oxen and 8 sheep cost 8 taels, what is the price of each?"[1]

A partnership problem is the following:

"When buying things in companionship, if each gives 8 pieces, the surplus is 3; if each gives 7, the deficiency is 4. It is required to know the number of persons and price of the things bought."[2]

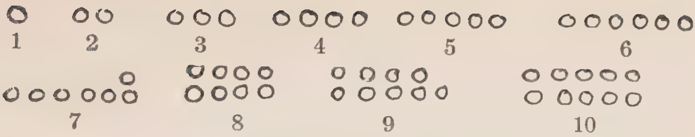
Oriental source of European problems. Unfortunately the date of composition ascribed by Chinese scholars, the first century B.C., appears to be based upon insufficient evidence. In any event, however, the Chinese had a real gift for numerical problems quite analogous to that displayed by the Hindus and Arabs. The oriental source of many problems which appeared in Europe in 1202 in Leonard of Pisa's voluminous work on arithmetic is not to be denied. Not only the same type of problems as in the early Chinese and Hindu works are given by Leonard, but frequently precisely the same series of numbers, so that the oriental origin is evident. These problems were taken over by Italian arithmeticians and then from them by other Europeans. By this route the problems of ancient India and China found their way into American textbooks.

Native American Numeral Systems

Maya twenty system. The Mayas of Yucatan had a highly developed twenty system. The Mayas had separate words for 20, 400 or 20×20, and for 8000 or $20 \times 20 \times 20$.

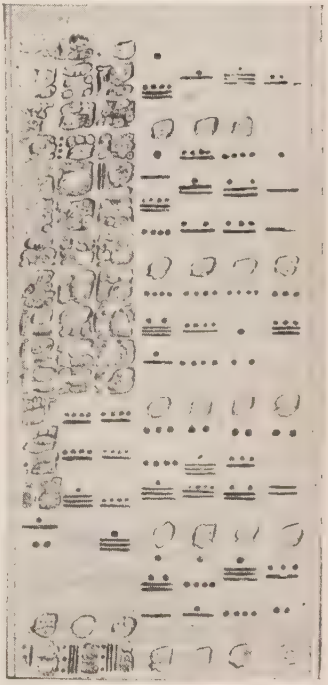
[1] Smith and Mikami, *A History of Japanese Mathematics* (Chicago, Open Court Pub. Co., 1924), p. 13.

[2] Mikami, *The Development of Mathematics in China and Japan* (Leipzig, Teubner, 1913), p. 16.



AZTEC NUMERALS

Primitive American numeral forms as found among the Aztecs in Mexico. (*See* the frontispiece.)



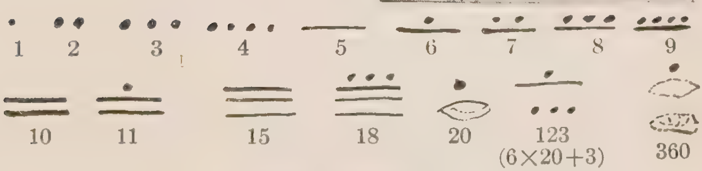
MAYA CODEX

In column form in the lower right-hand corner is found the Maya numeral representing 8 years; 8 in the third line represents 8×360; 2 in the second line represents 2×20 or 40; together this makes 8×(360+5) or 8×365.

At the left this is doubled, giving 16:4:0; then added again, giving 24:6:0 which is written 1:4:6:0, in a vertical column.

In the next numeral three dots were left out of the original, which should be 1:12:8:0. This continues above at the right.

The other symbols represent months and days.



MAYA NUMERALS

The Maya symbols proceed by powers of 20 except the third place, which represents not 20×20 but 18×20 or 360, which was more convenient for calendar purposes.

In some early Mexican languages the word for 20 is "man"; for 10 it is "two hands," and for 5, "one hand."

MAYA HIEROGLYPHIC NUMBER SYMBOLS

The flag represents 20; thus 200 jars of honey are indicated. The spiked leaf represents 400; 2000 blankets are indicated; 400 covered baskets; 1200 open baskets. The epaulet-like symbol at the bottom of the second column stands for 8000 or $20 \times 20 \times 20$; 8000 pellets of copal, used as incense, are indicated. All from the Mayan Codex of Mendoza, now in the Royal Library at Dresden.

The Mayas paid a great deal of attention to the calendar, and so most of their numerals are in connection with months and days and years. The year consisted of eighteen months of twenty days each; five extra holidays were included at the end of each year.

Maya picture numerals. The Maya picture numerals suggest the hieroglyphics; indeed, their writing is probably a picture writing. Twenty baskets are represented by a basket with a flag flying from it; the flag is the symbol for 20. For 400 the symbol is a spiked leaf; for 8000 another symbol, representing something like an epaulet.

PERUVIAN "QUIPU"

This numerical record represents probably a census of some district. Occasionally variously colored cords were used to represent men, women, and children respectively. Each cord here may represent some district or family. The final sum commonly appears on a single major strand.

The writer is indebted to Dr. L. Leland Locke for the illustration.

Peruvian knots. The Peruvians used knots upon strings of different colors. A large knot represented ten; small knots represented units. This instrument is called a *quipu*. The American Indians did not live in great cities like the Aztecs, Mayas, and Peruvians, nor did they reach as high a degree of civilization. Hence the Indians did not develop any very extensive systems of representation of numbers.

European Developments of the Abacus

"Reckoning on lines." The use of the abacus led to another type of computation, called *reckoning on lines*. We sometimes speak today of "casting an account," which refers, as we shall see, to this type of representation of numbers.

Little round markers or counters were made to use upon lines drawn upon a table called in German *ein*

Rechenbuch/
Auff Linien vnd Ziffern/ für die junge angehende Schüler. Mit einem leichten Visirbüchlein/ Klar vnd verstendlich fürgeben.
Gerechnet Büchlin/ auff alle Wahr vnd Kauffmanschafft/ Müntz/ Gewicht/ Elen/ vnd Maß/ vier Land vnd Stett verglichen.
Durch H. Jacob Köbel.
Cum Gratia & Priuilegio.
M. D. LXXXIIII.

The popular German work by Köbel, *Arithmetic upon Lines and with Numerals*. At the right and at the back lines are being employed; the man at the left is using the pen as opposed to counters.

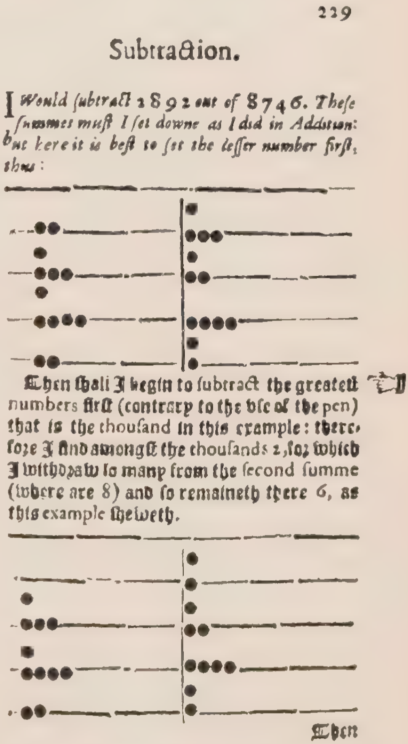
On the wall the numerals are represented on a kind of horn book similar to that used in early England and in colonial America.

Rechenbanck or simply *Banck*, from which we get our words "bank" and "banker," possibly through French mediation. The word "bankrupt" means a "broken board," the instrument used by the banker being actually broken.

A counter placed upon the first line represented a unit; placed on the second line a ten; and so on. A counter placed in any space represents five counters in the line just below the space. On the boards in the illustration

from Recorde are represented the numbers 2892, 8746, 892, and 6746 respectively.

In French the counters were called *jetons*, from the French verb *jeter*, meaning "to throw." In Latin treatises the counters were called *projectilia;* in Germany the markers were called *Rechenpfennige.* In England a kind of checkered board was used for "casting accounts,"


229

Subtraction.

I Would subtract 2892 out of 8746. These summes must I set downe as I did in Addition: but here it is best to set the lesser number first, thus:

Then shall I begin to subtract the greatest numbers first (contrary to the use of the pen) that is the thousand in this example: therefore I find amongst the thousands 2, for which I withdraw so many from the second summe (where are 8) and so remaineth there 6, as this example sheweth.

Then

Robert Recorde's explanation of subtraction, employing counters. At the right is represented 8746; 1 is on the lowest line and 5 in the space above it, represented by one counter there; four counters on the second line give 40; two on the third line for 200, with one above for 500; three counters on the fourth line for 3000, and a counter in the space above for 5000.

At the left, 2892; below, the first step in the subtraction has been performed.

from which we get the words "exchequer" and "check" or "cheque," and the expression "to check an account."

The "reckoning on lines" began in the thirteenth century and extended over all of Europe. Long after the invention of printing treatises on this subject continued to appear, usually together with the written arithmetic. The two methods were frequently contrasted in illustrations in early arithmetics as in the title page of Köbel's arithmetic of 1584 on page 34.

The details of the operations involve largely the inevitable peculiarities due to the type of notation. Undoubtedly the work seems much more awkward to us than it actually was to one experienced with this form of representation of numbers. The long-continued use of the abacus and of this visual form of representation of numbers testifies to the usefulness of tangible and visual aids in instruction. Teachers do well to use such methods wherever possible in instruction.

Bibliography for Supplementary Reading

T. Eric Peet, *The Rhind Mathematical Papyrus.* Liverpool, University Press, 1923.

H. V. Hilprecht, *Mathematical, Meteorological and Chronological Tablets from Nippur.* University of Pennsylvania Studies, Vol. 19.

T. L. Heath, *A History of Greek Mathematics.* 2 vols. Oxford, 1921.

J. F. Payne, "Natural History and Science," in *A Companion to Latin Studies,* edited by Sir J. E. Sandys. Cambridge, 1921. Third edition.

F. P. Barnard, *The Casting-Counter and the Counting Board.* Oxford, Clarendon Press, 1916.

D. E. Smith, *Computing Jetons.* New York, American Numismatic Society, 1921.

Yoshu Mikami, "The Development of Mathematics in China and

Japan." *Abhandlungen zur Geschichte der Mathematischen Wissen.* Vol. XXX. Leipzig, Teubner, 1913.

D. E. Smith and Y. Mikami, *A History of Japanese Mathematics.* Chicago, Open Court Pub. Co., 1914.

C. P. Bowditch, *The Numeration, Calendar Systems and Astronomical Knowledge of the Mayas.* Cambridge, University Press, 1910.

L. Leland Locke, *The Ancient Quipu or Peruvian Knot Record.* New York, American Museum of Natural History, 1923.

Cyrus Thomas, "Mayan Calendar Systems." I: U. S. Bureau of Ethnology, 19th *Report* (Washington, 1900), pp. 693–820; II: U. S. Bureau of Ethnology, 22d *Report* (Washington, 1904), pp. 197–305.

Cyrus Thomas, "Numeral Systems of Mexico and Central America." U. S. Bureau of Ethnology, 19th *Report*, pp. 853–955.

W. J. McGee, "Primitive Numbers." U. S. Bureau of Ethnology, 19th *Report*, pp. 821–851.

See encyclopedias under Babylon, Cuneiform, Egypt, Greece, Hieroglyphic, Maya, Mexico, Peru.

CHAPTER II

THE NUMERALS WHICH WE USE TODAY

Hindu Origin

"And Viswamitra said, 'It is enough,
Let us to numbers.
After me repeat
Your numeration till we reach the Lakh,
One, two, three, four, to ten, and then by tens
To hundreds, thousands.' After him the child
Named digits, decads, centuries; nor paused,
The round lakh reached, but softly murmured on,
'Then comes the kōti, nahut, ninnahut,
Khamba, viskhamba, abab, attata,
To kumuds, gundhikas, and utpalas,
By pundarikas unto padumas,
Which last is how you count the utmost grains
Of Hastagiri ground to finest dust;
But beyond that a numeration is,
The Kātha, used to count the stars of night;
The Kōti-Kātha, for the ocean drops;
Ingga, the calculus of circulars;
Sarvanikchepa, by the which you deal
With all the sands of Gunga, till we come
To Antah-Kalpas, where the unit is
The sands of the ten crore Gungas. If one seeks
More comprehensive scale, th' arithmic mounts
By the Asankya, which is the tale
Of all the drops that in ten thousand years
Would fall on all the worlds by daily rain;
Thence unto Maha Kalpas, by the which
The gods compute their future and their past.' "

. . . .

. . . . " 'And, Master! if it please,
I shall recite how many sun-motes lie

From end to end within a yōjana.'
Thereat, with instant skill, the little Prince
Pronounced the total of the atoms true.
But Viswamitra heard it on his face
Prostrate before the boy; 'For thou,' he cried,
'Art Teacher of thy teachers — thou, not I,
Art Gūrū.' "[1]

Reading of large numbers. Centuries before the Christian Era, Hindu writers showed a great fondness for calculating with large numbers. The Hindus carried the decimal numeration, naming of the successive powers of ten, far beyond that of any other people. In different parts of India varying names were used for some of the higher powers, but knowing the complete sequence the value of any unit was given by its *place* in the sequence; thus with units, tens, hundreds, thousands, the fourth name applies to the third power of 10, one thousand equals 10^3. The seventh name applies to the sixth power of ten, and similarly with every other name. In reading a large number in which every place is represented the Hindu read the name of each place in turn as indicated in the verses at the beginning of this chapter. In our reading of numbers we follow the early Arabs and the later Germans in grouping with reference to thousands and powers of one thousand; the Greeks grouped by myriads or ten thousands; the Hindu way calls attention to the *place*, or sequence value.

A large number, like 8,443,682,155, is read according to these different systems in the English, Sanskrit, Arabic

[1] This quotation from Sir Edwin Arnold's *The Light of Asia* indicates the prominent place given to arithmetic and numbers in the education of the Buddha. The passage suggests the problem discussed by Archimedes to indicate the number of grains of sand in the seashore.

and early German, and in the Greek language as follows:

English: 8 billion, 443 million, 682 thousand, 155.

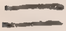
Sanskrit or Hindu: 8 padmas, 4 vyarbudas, 4 kōtis, 3 prayutas, 6 laksas, 8 ayutas, 2 sahasra, 1 śata, 5 daśan, 5.

Arabic and early German: Eight thousand thousand thousand and four hundred thousand thousand and forty-three thousand thousand, and six hundred thousand and eighty-two thousand and one hundred fifty-five (or five and fifty).

Greek: Eighty-four myriads of myriads and four thousand three hundred sixty-eight myriads and two thousand and one hundred fifty-five.

When every place is present we would know the number if we read (beginning with the units), simply: five, five, one, two, eight, six, three, four, four, eight. When certain powers of ten are not present, as in 3,080,046, we could read the number six, four, vacant, vacant, eight, vacant, three.

Development of the zero. This idea of using a word and finally a symbol for a vacant order of numbers was most highly developed in India. However, quite early in Greece the initial letter of the Greek word for "vacant" was occasionally used in writing degrees, minutes, and seconds; in Babylon a zero symbol  was introduced several centuries before the Christian Era. Even the Mayas of Yucatan had a similar idea with their twenty system. But only with the Hindus was the idea carried to its full logical development, to a place system of numbers in which any number, however great, can be expressed with the symbols for the first nine integers and a zero, and with all computations reduced to combinations indicated by these symbols. To make the system easily

applicable to computation it is essential that the nine unit symbols in a decimal system, or the nineteen in a twenty system, should be independent like the letter symbols, and not compounded. Thus  in Babylonian symbols or  in Maya represent 2 and 10; not 61 and 105 (5 twenties and five) as they would in a pure place system. Each of these symbols occupies two places instead of one.

The Sanskrit word for vacant is *śūnya*, and this word was used in numeration with this idea. Later a symbol was developed for this, a dot or a small circle. The Arabs about 800 A.D., in arithmetical treatises, translated *śūnya*, writing the Arabic word *sifr*, meaning vacant. This word was transliterated about 1200 A.D. into Latin, the sound and not the sense being kept, becoming *cyfra* and *tziphra* and *zephirum*. The difficulties in transliteration from a different alphabet are indicated by the "c," "tz," "z," "ch" as in *chiffre*, "ç" in *çero*, and "s" in *sifr*, all to represent the same Arabic letter. Various progressive changes of these forms have given us our words, "cipher" and "zero." In early English and American schools "ciphering" meant computation; in French the word for digits is *chiffres*, and similarly in some other of the Romance languages. This double meaning of "cipher" appears also in the early printed works explaining our numerals, but not in the original Hindu works.

Hindu word and letter systems. The Hindus were accustomed to put history and even astronomy and mathematics into verse. For the sake of the rime it was necessary to have several alternate ways of giving the numbers which were involved in the verses. The Hindus developed, probably before 600 A.D., a word-system of

recording numbers in which the place idea is prominent. For one was used "Buddha," "sun," or "moon"; for two was used "twins" (*yama*), "eyes" (*nayana*), or "hands" (*kara*); "oceans" for four; "senses" (*visaya*) or "arrows" (the five arrows of Kāmadēva) for five; seven by "mountain" (*aga*); eight by *Vasu*, the eight gods; and so on. In this system zero is given by "point" or "vacant" (*śūnya*), or by "heaven-space" (*ambara*, *ākāśa*). To give the number two thousand eight hundred and five, the words senses, vacant, Vasu, twins could be used; or arrows, heaven-space, Vasu, hands; note that the units are given first.

Another similar play on the place idea is found in a letter system used in southern India. The illustration will be with our letters, although the Sanskrit is better for this purpose, as it contains more consonants. In this system the consonants are given the values from 1 to 9 and 0 in turn, as follows:

b	*c*	*d*	*f*	*g*	*h*	*j*	*k*	*l*	*m*
1	2	3	4	5	6	7	8	9	0
n	*p*	*q*	*r*	*s*	*t*	*v*	*w*	*x*	*y*
1	2	3	4	5	6	7	8	9	0
z									
1									

To give 1492 by words, you pick out of the consonants for 1, 4, 9, and 2, a group such that by inserting only vowels you get full words. Thus "near lace" or "by real ice" would represent 1492. Apparently those Hindus who used this system had a great deal of time to spend on thinking up words.

1	4	9	2
b	*f*	*l*	*c*
n	*r*	*x*	*p*
z			

Modern Sanskrit numerals. The Sanskrit numerals as used now in India are of the following form:

१ २ ३ ४ ५ ६ ७ ८ ९ ०

1 2 3 4 5 6 7 8 9 0

MODERN SANSKRIT NUMERALS

An arithmetical work on bark, probably a thousand years old, found in India, contains numeral forms analogous to the Sanskrit. Although the date is somewhat uncertain it is without serious question the oldest Hindu arithmetical document extant. Included are a large number of problems requiring algebraic processes for their solutions and similar to later Arabic and European problems, particularly like many in Leonard of Pisa's work of 1202 A.D. The fraction forms are suggestive of Arabic. One problem reads:

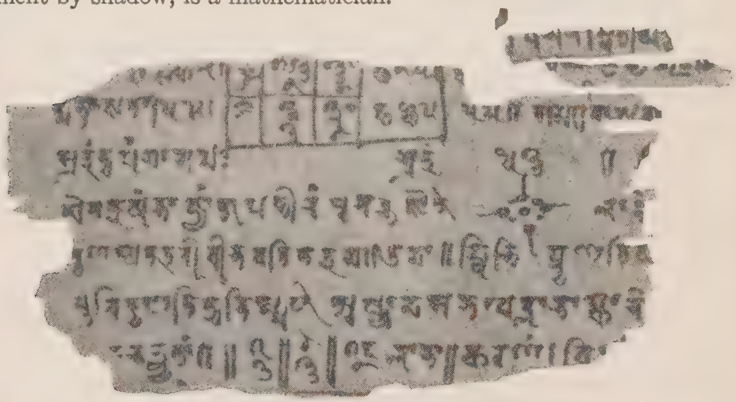
"One who purchases 7 for 2, sells 6 for 3. 18 is his profit. Say now, what was his capital?"[1]

Hindu processes and problems. The details of the fundamental operations as practiced by the Hindus will be touched upon in the later systematic discussion of the fundamental operations. No early Hindu treatise gives us detailed and clear accounts of the processes of the arithmetical operations. Hence we are not able to connect the methods of Arabic arithmeticians directly with Hindu sources to which the Arabs give credit. However, the explanations as far as given, and more particularly the topics and the content of the Hindu arithmetic, correspond to the work of the Arabs.

[1]Rudolph Hoernle, "The Bakhshali Manuscript," *Indian Antiquary*, Vol. XVIII (1888), pp. 33–48, 275–279, three plates.

Even as early as Brahmagupta (seventh century A.D.) the systematization of arithmetic had attained to a high stage of development. Brahmagupta states:

"He, who distinctly and severally knows addition and the rest of the twenty logistics, and the eight determinations including measurement by shadow, is a mathematician."[1]



BAKHSHALI ARITHMETICAL MANUSCRIPT

The earliest Hindu manuscript on arithmetic employing the numerals with place value. The symbols within the checkered diagram represent:

	13 6	30 1	
1 1	3 2	30 1	45

A later commentator gives the list as follows:

"Addition, subtraction, multiplication, division, square, square root, cube, cube root, five (should be, six) rules of reduction of fractions, rule of three terms (direct and inverse), of five terms, seven terms, nine terms, eleven terms, and barter, are twenty arithmetical operations. Mixture, progression, plane figure, excavation, stack, saw, mound, and shadow are eight determinations." [2]

[1] H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bháskara* (London, 1817), p. 277.

[2] Colebrooke, *loc. cit.*, p. 277.

Mahavir's arithmetic. The treatise by Mahavir some two centuries later (*c.* 925 A.D.) takes largely similar topics, but the number and variety of the numerical illustrations is greatly increased. Mahavir states the topics of arithmetic most engagingly, as follows:

"With the help of the accomplished holy sages, who are worthy to be worshiped by the lords of the world, and of their disciples and disciples' disciples, who constitute the well-known jointed series of preceptors, I glean from the great ocean of the knowledge of numbers a little of its essence, in the manner in which gems are (picked up) from the sea, gold is from the stony rock and the pearl from the oyster shell; and give out, according to the power of my intelligence, the *Sarasangraha*, a small work on arithmetic which is not small in value.

"Accordingly, from this ocean of *Sarasangraha*, which is filled with the water of terminology and has the arithmetical operations for its bank; which is full of the bold rolling fish represented by the operations relating to fractions, and is characterized by the great crocodile represented by the chapter of miscellaneous examples; which is possessed of the waves represented by the chapter on the rule-of-three, and is variegated in splendor through the luster of the gems represented by the excellent language relating to the chapter on mixed problems; and which possesses the extensive bottom represented by the chapter on area problems, and has the sands represented by the chapter on the cubic contents of excavations; and wherein shines forth the advancing tide represented by the chapter on shadows, which is related to the department of practical calculation in astronomy — (from this ocean) arithmeticians possessing the necessary qualifications in abundance will, through the instrumentality of calculation, obtain such pure gems as they desire."[1]

Bháskara's arithmetic. The following problems from Bháskara's *Lilavati* of the twelfth century have a truly familiar sound:

[1] M. Rangacarya, *The Ganita-Sara-Sangraha of Mahaviracarya*, with English translation and notes (Madras, 1912), pp. 3–4.

"Say quickly, friend, in what portion of a day will (four) fountains, being let loose together, fill a cistern, which, if severally opened, they would fill in one day, half a day, the third, and the sixth part, respectively?"[1]

"If three and a half *mánas* (a measure) of rice may be had for one *dramma*, and eight of kidney beans for the like price, take these thirteen *cácinis*, merchant, and give me quickly two parts of rice with one of kidney beans; for we must make a hasty meal and depart, since my companion will proceed onwards."[2]

Mahavir has numerous problems resulting in strings of units or zeros, with a kind of play on the zero. Thus the problems:

"In this (problem) write down 3, 4, 1, 7, 8, 2, 4, and 1 (in order from the units' place upwards), and multiply by 7; and then say that it is the necklace of precious gems." *Ans.* 100,010,001.

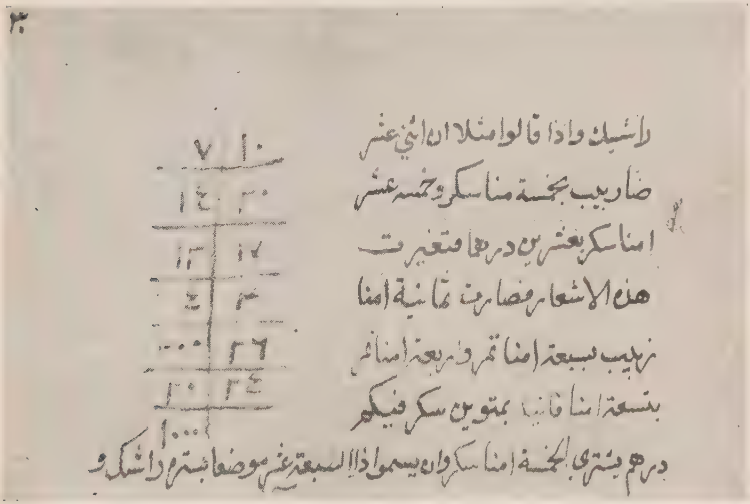
"Write down (the number) 142857143, and multiply it by 7; and then say that it is the royal necklace." *Ans.* 100,000,001.[3]

Hindu interests. Not only the topics but the methods and terminology of arithmetic are highly developed in all of the early Hindu arithmetical works. In all fundamental aspects the Hindu arithmetic corresponds to the modern subject much more closely than the same subject as developed by any other people before the year 800 A.D.

Delight in computation in and for itself is evident throughout the Hindu arithmetics. This love for computation led them to expand the subject of business arithmetic by inverse problems under which our children continue to suffer even today. But the love of computing led the Hindu into other problems of algebraic and

[1]Colebrooke, *loc. cit.*, p. 42.
[2]Colebrooke, *loc. cit.*, p. 43.
[3]Rangacarya, *loc. cit.*, p. 11.

trigonometric nature. These developments were destined to play a great part in the further progress of mathematical science.



ARABIC MANUSCRIPT ON ARITHMETIC

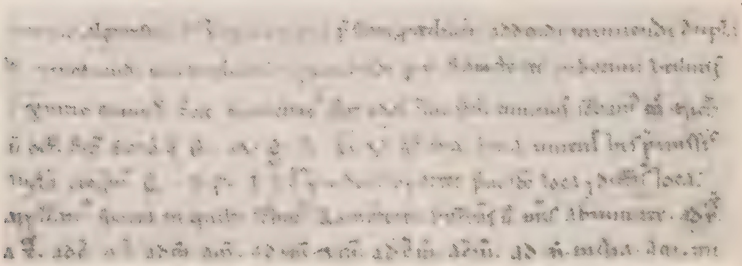
Arabic discussion of problems involving proportion by Albiruni, a famous Arab of the tenth century A.D. At the left the numbers 7, 15, 13, 5, 300, 20, and 1000 in the first column; the numbers 10, 20, 17, 4, 26, and 24 in the second column.

This photograph was obtained through the courtesy of Dr. Karl Schoy of Essen.

The Arabic Development of the Hindu Arithmetic

Arabic learning. About 800 or 825 A.D., or possibly fifty years earlier, the Arabs learned the details of the Hindu system of arithmetic. At the same time the Arabs were diligently studying the astronomy of the Hindus and the scientific works of the Greeks. The Arabs were then and continued for six centuries the most serious

students in the world; they kept the torch of learning aglow while Europe was in darkness. Several Arabs in



MS. Royal, 15 B IX, British Museum

TWELFTH-CENTURY ALGORISM IN LATIN

This is part of a single page containing a complete discussion of the new numerals, with the fundamental operations.

The opening line reads: "Intencio Algorismi est (÷) in hoc opere doctrinam praestare, procedendi, addendi, minuendi duplan-"; the second line continues: "-di, et mediandi multiplicandi et diuidendi per x karacteres indorum."

In the fourth line the numerals appear: 0, 9, 8, 7, 6, 5 (Roman V) and below 4, 3, 2, 1.

As in most manuscripts of this period, many abbreviations are used; the symbol like our division sign stands for *est*, a line through the stem of a "p" makes it *per*, while a line over a letter usually stands for "n."

the ninth century wrote treatises on the Hindu art of reckoning. The earliest reference outside of India to the numerals which we now use is by a famous Syrian monk, Severus Sebokht, living in a monastery at Quenesre on the banks of the Euphrates. This Syrian was attempting to show that all science was not due to Greece. He said: "I will not now speak of the science of the Hindus . . . and of the easy method of their calculations and of their computations which surpasses words. I mean that made with nine symbols."[1]

Early Arabic arithmetics. The earliest systematic treatise on the new arithmetic which has come down to us is

[1] Karpinski, "The Hindu-Arabic Numerals," *Science*, 1912, Vol. 35, pp. 969–970.

that by Al-Khowarizmi, a Persian scientist who lived in the ninth century of the Christian Era. Even his work has not been found in Arabic, but is preserved only in a Latin translation made quite certainly in the twelfth century.

The English word "algorism" comes from the Latin form of his name. This word was long used to mean arithmetic with the Hindu-Arabic numerals; in French the form of the word was *augrim*, and this form of the word is used by Chaucer about 1350 A.D. and appears in English literature from the thirteenth century. The Latin translation of Al-Khowarizmi's arithmetic begins, "Dixit algoritmi"; this word *algorithm* later came to be used as title for the subject.

Arabic and Hebrew arithmetics. Numerous Arabic works on the new arithmetic appeared from the ninth to the fifteenth centuries. Several of these have been translated from Arabic into modern languages. The Arabs were good traders, so that the practical applications of the Hindu arithmeticians particularly appealed to them. The Arabs systematized their knowledge, making excellent textbooks on arithmetic and algebra which continued to influence European mathematics for several centuries.

Jewish students in Spain were also quick to learn the new numerals. Jewish and Arabic writers show greater appreciation of the possibilities of the new arithmetic than the writers of the early European treatises. Among the famous Jewish arithmeticians was the Rabbi Ben Esra (Ibrahim ibn Ezra), whose fame was sung by Robert Browning; his treatise on arithmetic, preserved

in a Hebrew manuscript, has been published with a German translation.[1]

Several of these early Arabic arithmetics have been preserved among the great European collections of oriental manuscripts. The Arabic arithmetics of Al-Nasawi[2] and Al-Karkhi[3] of the eleventh century A.D., of Al-Hassar[4] of the twelfth, of Ibn Al-Benna[5] of the thirteenth, and of Al-Kalçadi[6] of the fifteenth century have been made accessible in translation and a few others have been summarized.

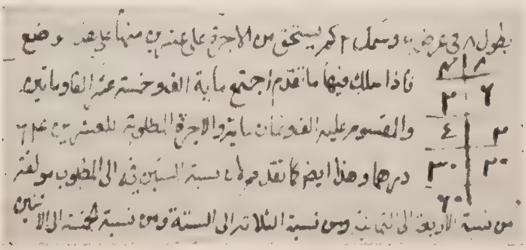
A twelfth-century Arabic arithmetic. The introduction to the work of Al-Hassar is characteristically Arabic: "In the name of God, merciful and compassionate. My Lord! Make easy (my task), oh thou Beneficent One. Speaks the teacher Abu Zakarija Mohammed Abdallah, known by the name Al-Hassar. Praise be to God, etc. . . . ." Al-Hassar continued with further calls upon Allah and also with interjected statements concerning the dependence of his work upon the writings of older scientists.

Integers and fractions are treated in separate sections. The work on integers falls under ten subheads: numeration, notation, addition, subtraction, multiplication, denomination including the check by nines, division, halving, doubling, and extraction of roots. The fractions involve numerous complications peculiar to the Arabs

[1] Silberberg, *Das Buch der Zahl des R. Abraham ibn Esra.* Frankfort, 1895.
[2] Woepcke, *Journal Asiatique*, I 6, pp. 491–500.
[3] Hochheim, *Programm.* Halle, 1878–1879.
[4] Suter, *Bibliotheca Mathematica*, Q 3, pp. 12–40.
[5] A. Marre, *Atti dell' Accad. Pont. de nuovi Lincei*, XVII, pp. 289–319.
[6] Woepcke, *Atti dell' Accad. Pont. de nuovi Lincei*, XII, pp. 230–275, 399–438.

which fortunately found little favor with their European translators.

Arabic shortcuts. An interesting innovation is found in Al-Karkhi's *Kafi fil Hisab*, or "Sufficient concerning Arithmetic" (*c.* 1010 A.D.). The author gives a number of shortcuts in multiplication when the multiplier is either an aliquot part of 100 or 1000 or near to such an aliquot part. In this text we find considerable material on mensuration of surfaces and solids, algebra with applied problems, but few applied arithmetical problems.



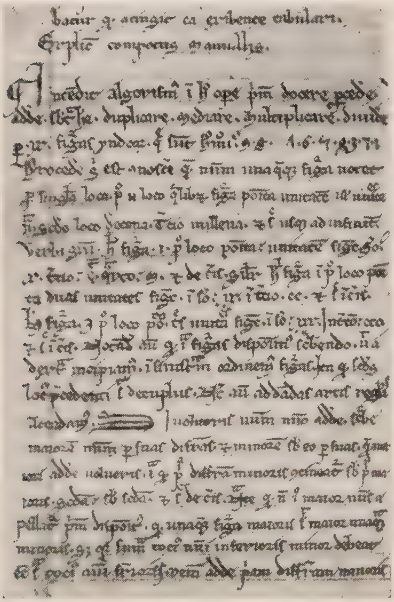
ARABIC MANUSCRIPT ON HINDU ARITHMETIC

This is a section from a treatise on compound proportion by the Arabic scientist and traveler, Albiruni, written about 1000 A.D. In the first column are the numerals 4, 2, 5, 30, 60; at the extreme right, 8, 6, 2, 20.

Arabic business arithmetic. A more complete business arithmetic was written by Abu 'l Wefa (990–998 A.D.) which unfortunately has not yet been published. This work in seven sections of seven chapters each is entitled "Treatise of that which is necessary in regard to calculation for collectors and clerks." The chapter headings indicate a comprehensive commercial arithmetic with mensuration, exchange, and denominate numbers. The one striking omission in his work appears to be the subject of interest.

European Translators of Arabic Scientific Works

The Arabs in Spain. The Arabs entered Spain in the year 772 A.D. and continued in parts of Spain until shortly before the discovery of America. During all of this time there was much contact between the Arabs and the



MS. Egerton 2261, British Museum

THIRTEENTH–CENTURY ALGORISM

End of a work on the Computus (on the calendar) and beginning of an early explanation of our numerals.

The first three lines of the algorism read: "Intendit algorismus in hoc opere primum docere, procedere addere, subtrahere, duplicare, mediare, multiplicare, diuidere per ix figuras yndorum que sunt huius modi 0 9 8 7 6 5 4 3 2 1."

In the first line only one word is written out in full; all of the others have the abbreviations characteristic of Latin manuscripts made during the three or four centuries preceding the invention of printing.

Europeans. At the beginning of the twelfth century the fame of the Moslem schools at Toledo in Spain and in other Spanish cities had spread over all of Europe. The Crusades, also, aroused new interest in the Arabs and their teachings. In the twelfth century activity in the translation into Latin and Hebrew of Arabic scientific works reached its highest point. The greatest of the

translators was Gerard of Cremona, an Italian. This student of astronomy heard that there was a copy of the great Greek work on astronomy, Ptolemy's *Almagest*, to be had in Spain. He made the long and difficult journey from Italy to Spain in order to get this work. When he arrived in Spain he found the work, but it was in Arabic. Not to be daunted, Gerard with the help of a Jewish scientist who knew Arabic and Spanish, in which language they probably conversed, translated the *Almagest* into Latin. While doing this he learned of the great number of Arabic works of science, and he determined to devote his life to the translation into Latin of such works. Gerard spent nearly fifty years in Spain, translating medical, astronomical, philosophical, and mathematical treatises.

Latin translations from Arabic. The earliest European treatises on the new arithmetic and on algebra were Latin translations made during the twelfth century. Robert of Chester, an Englishman, made one translation of the algebra of Al-Khowarizmi; another translation of the algebra was probably made by Gerard of Cremona. The arithmetic by the same author was twice translated in the twelfth century, possibly by Adelard of Bath and John of Luna, a Spaniard. Several other Latin treatises directly related to the translations appeared during the twelfth century.[1] The total number of complete explanations of the new arithmetic written before 1300 A.D. and available today probably does not exceed twenty, published and still in manuscript.

The early translations were exceedingly concise and required considerable explanation for people who were

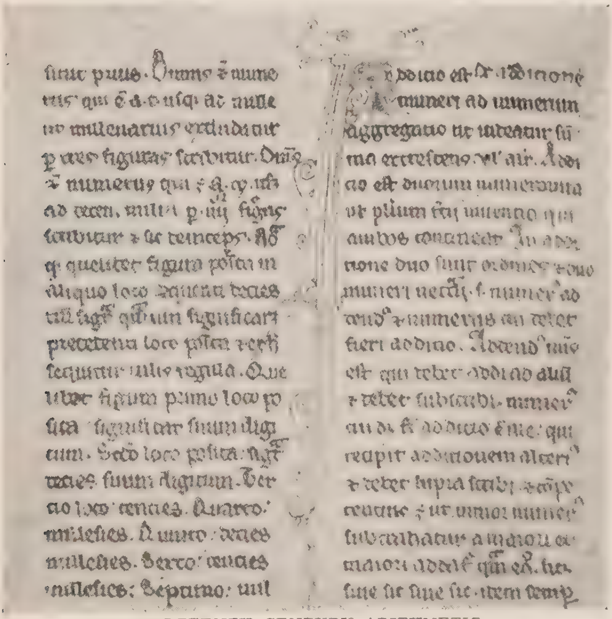
accustomed only to the Roman (rarely to Greek) numerals and the abacus. Early in the thirteenth century at least three more extended treatises on arithmetic appeared. This subject became a part of the curriculum in the universities which were just being formed in France, Italy, England, and Germany.

Thirteenth-century arithmeticians. The *Algorismus Vulgaris* (common algorism) of John of Holywood or Halifax or Sacrobosco was the most widely used of these three treatises, and copies of it written by students of mathematics in the thirteenth to fifteenth centuries are found in many European libraries. The *Liber Abbaci* of Leonard of Pisa was the most extensive treatment; it was printed in Italy in 1852, more than six hundred years after it was written, and the sections on arithmetic cover some two hundred pages. The length explains why it never became popular. A Frenchman, Alexandre de Ville Dieu, wrote a treatise in Latin verse, *Carmen de Algorismo*, which was second in popularity only to Sacrobosco's *Algorismus*. A German, Jordanus Nemorarius, wrote an explanation of the new arithmetic with demonstrations in Euclidean form, *Demonstratio Jordani in Algorismo*, but it did not become widely popular.

All of these works were in Latin, which was the language of the universities and the language in which European scientific works were commonly written up to the eighteenth century. Three of the works cited above have the word *algorism* involved in the title, and Leonard of Pisa also uses the word.

In the thirteenth and fourteenth centuries there were thousands of students in the European universities who

became familiar with the new arithmetic. Commonly the teacher would read Sacrobosco's treatise, line by



THIRTEENTH–CENTURY ARITHMETIC

This is Sacrobosco's *Algorismus* as found in Codex Arundel 332, British Museum.

The decorated column has the heading at the right, "De addicione." The text of this reads: "Addicio est numeri ad numerum aggregatio . . . . " In the eighth line note the symbol like a + sign for *et* or "and."

line, explaining each line as he went along. One of these extended commentaries made in 1292 by Petrus de Dacia, lecturing at Paris, has been published. The commentary covers about four times as many pages as does the text, which occupies some twenty printed pages.

Early English Arithmetic

Early treatises in the vulgar tongues. Latin continued until well into the eighteenth century as the language of instruction in European schools. However, occasionally treatises appeared in the vulgar tongues. The new arithmetic appeared in the fourteenth century in Icelandic, a translation of Sacrobosco; in the fourteenth century a discussion based on the *Carmen* by Alexandre de Ville Dieu appeared in French; in the next century a German treatise appeared.

In English brief and incomplete discussions appeared in the fourteenth century, but no complete work is known earlier than the two fifteenth-century treatises discussed below. Even in general English literature of the thirteenth and fourteenth centuries references to the new arithmetic are extremely rare. The *Ancren Riwle* refers to the "nombres of augrym" and Langlois, in *Rechard Redeles* of 1399, mentions: "as siphre doth in awgrym, that noteth a place and no thing availith."

First English arithmetics. *The Crafte of Nombrynge* (fifteenth century) is found in a single manuscript, Egerton MS. 2622, in the British Museum.[1] This is a running commentary on the *Carmen de Algorismo*. It begins:

"This boke is called the boke of algorym, or Augrym after lewder vse . . . .

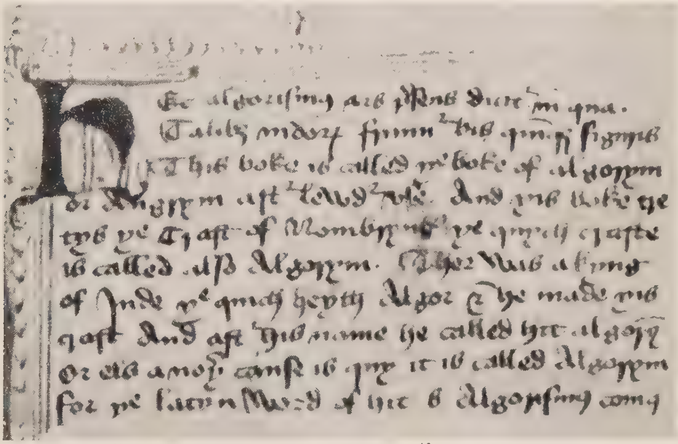
" . . . . fforthermore ȳe most vndirstonde that in this craft ben vsid teen figurys, as here bene writen for ensampul, ∅987654321."[2]

The other fifteenth-century treatise is a translation with commentary of Sacrobosco's *Algorismus Vulgaris*. This

[1] Recently published by Robert Steele, *The Earliest Arithmetics in English*, Oxford Univ. Press, 1922. [2] Steele, *loc. cit.*, p. 3.

Art of Nombryng is found in another of the great English repositories of manuscripts, the Bodleian Library at Oxford, MS. Ashmole 396. (*See* illustration, page 107.)

The opening words are: "Boys seying in the begynnyng of his Arsemetrike:—Alle thynges . . . . of nom-



"THE CRAFTE OF NOMBRYNGE," *c.* 1450 A.D.

In this earliest treatise on arithmetic in English the first two lines are in Latin, a quotation from the verses of Alexandre de Ville Dieu. The Latin begins: "Hec algorismus ars praesens dicitur . . . ." The English begins: "This boke is called the boke of algorym," etc. The last line reads: "for ye latyn word of hit is Algorismus comes."

bre ben formede." A little farther on the writer says: "So algorisme is clepede the art of Nombryng . . . ." "Boys" stands for Boethius, whose arithmetic, largely a translation of the Greek work of Nicomachus, was in wide use both before and after the introduction of the Hindu-Arabic numerals.

Printing of arithmetical works in England. In English the earliest printed discussion of arithmetic was published by the first English printer, William Caxton, in the first

English work containing illustrations, *The Mirrour of the World.* On the page discussing arithmetic all of the new

The fourth science is called arsmetrique this science cometh after rethoryque / & is sette in the myddle of the vii sciences / And wythout her may none of the vii sciences parfyghtly ne weel and entierly be knowen Wherfore it is expedyent that it be wel knowen & conned For all the sciences take of it their substaunce in suche wyse that wythout her they may not be / And for this reson was she sette in the myddle of the vii sciences. & there holdeth her nombre. For fro her procede al maners of nombres And in alle thynges renne comme & goo. And no thyng is wythout nombre. But fewe perceyue how this may be. But yf he haue be maystre of the vii Artes so longe that he can truly saye the trouthe But we may not now recompte ne declare alle the causes wherfore / For who that wolde dispute vpon suche werkes him behoued despute and knowe many thynges and moche of the glose who that knewe well the science of arsmetryque he myght see thordynaunce of alle thynes by ordynaunce was the world made and created. And by ordynaunce of the souerayn it shal be deffeted /

Next foloweth the science of Geometrye / Capitulo / xi°

From the famous Caxton Press, London, 1481

This brief discussion of arithmetic is based upon a thirteenth-century encyclopedic work by Vincent de Beauvais.

"The fourth science is called arsmetrique this science cometh after rethoryque, and is sette in the myddle of the vii sciences. And wythout her may none of the vii sciences parfyghtly ne weel and entierly be knowen."

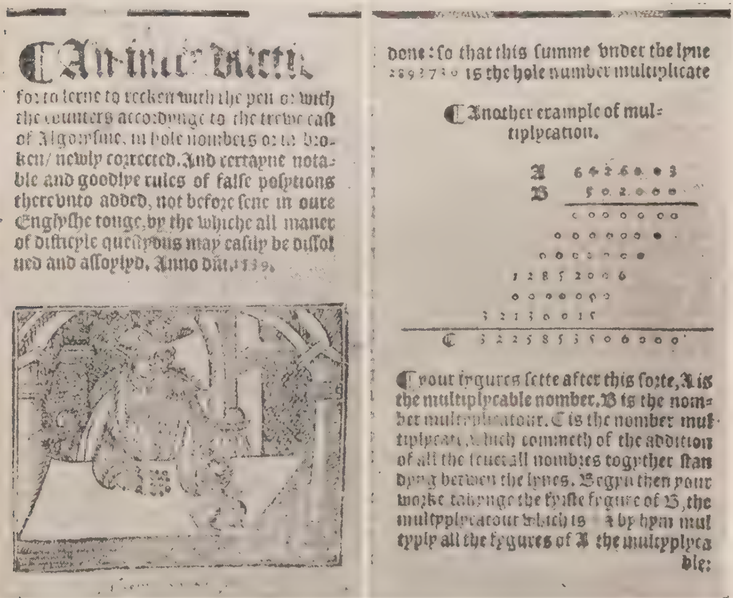
The numerals are indicated on the tablets or scrolls, and a heap of counters is on the table.

Caxton was the first printer in England.

forms, apparently not fully comprehended by the illustrator, appear on a kind of horn book, but they are not placed so as to make a rational problem; some forms are reversed. No reference is made to these forms in the text, nor is there anything except the most general praise of arithmetic.

"The fourth scyence is called arsmetrique . . . . Who that knewe wel the science of arsmetrique he myght see thordynaunce of alle thynges." The seven liberal arts

include grammar, logic, and rhetoric, together called the trivium; and arithmetic, "the fourth science," with geometry, music, and astronomy, called the quadrivium.



¶ An Introduction for to lerne to recken with the pen or with the counters accordynge to the trewe cast of Algorysme, in hole nombers or in broken/ newly corrected. And certayne notable and goodlye rules of false posytions therevnto added, not before sene in oure Englysshe tonge, by the whiche all maner of difficyle questyons may easily be dissolued and assoyled. Anno dñi. 1539.

done: so that this summe vnder the lyne is the hole number multiplicate

¶ Another example of multiplycation.

¶ Your fygures sette after this forte, A is the multiplycable nomber, B is the nomber multiplicatour. C is the nomber multiplycate, which commeth of the addition of all the seuerall nombres togyther standyng betwen the lynes. Begyn then your worke takynge the fyrste fygure of B, the multyplycatour which is & by hym multyply all the fygures of A the multyplycable:

ONE OF THE EARLIEST PRINTED ARITHMETIC TEXTS IN ENGLISH, 1539

There was printed in 1537 a somewhat similar work. Recorde's arithmetic appeared five years later and became popular.

The earliest printed works in English which explain the new numerals are two anonymous treatises of 1537 and 1539. The earlier one is entitled: "An Introduction for to lerne to reckon with the Pen and with the Counters after the true cast of Arsmetyke, or Awgrym"; it was published at St. Albans. The other work, published in London, is entitled, as may be read above: "An Introduction for to learne to recken with the pen, or with the

counters." The latter of these was published in a second edition in London in 1546, and there were later editions.

The English treatise which is primarily concerned with the popularization of our system of numerals and computation is Robert Recorde's *The Grounde of Artes*, which appeared first about 1542 and in twenty-seven further editions up to 1699. So far as America is concerned English texts were long imported, as well as Spanish, Dutch, French, and German. The first separate English text on arithmetic in the United States appeared in Boston in 1719, but it was preceded by Spanish works by more than a century.

Bibliography for Supplementary Reading

D. E. Smith and L. C. Karpinski, *The Hindu-Arabic Numerals.* Boston, Ginn and Co., 1911.

G. F. Hill, *The Development of Arabic Numerals in Europe.* Oxford, Clarendon Press, 1915.

G. R. Kaye, *Indian Mathematics.* Calcutta, 1915.

H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhàskara.* London, 1817.

M. Rańgacarya, *The Ganita-Sara-Sangraha of Mahaviracarya.* Madras, Government Press, 1912.

L. C. Karpinski, "Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi." University of Michigan, *Humanistic Series*, Vol. XI, Part I. New York, 1912.

Augustus De Morgan, *Arithmetical Books from the Invention of Printing to the Present Time.* London, 1847.

J. O. Halliwell, *Rara Mathematica; or, A Collection of Treatises on the Mathematics.* London, 1839.

D. E. Smith, *Rara Arithmetica.* Boston, Ginn and Co., 1908. Bibliography of printed arithmetics to 1601 with special reference to those in the library of Mr. George A. Plimpton in New York.

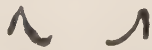
Charles H. Haskins, *Studies in the History of Medieval Science.* Cambridge, Harvard University Press, 1924.

CHAPTER III

THE TEXTBOOKS OF ARITHMETIC

Egyptian

Egyptian textbook. The first systematic treatise on mathematics is the Ahmes papyrus, which represents the type of instruction given in Egypt nearly four thousand years ago. The work was evidently designed as a textbook; the occasional use of red ink suggests, in fact, a modern teacher's corrections. Both the problems and the methods employed in this ancient manual continued to appear in Egypt for centuries, in Greek arithmetic up to 1000 A.D., and even in Latin treatises of the thirteenth century. Between the Egyptian work and the Greek treatises on arithmetic nearly fifteen hundred years intervene. The separation of arithmetical material and geometrical material on mensuration into a single work as done by the Egyptians constitutes a notable step in the progress of science and civilization.



SYMBOLS FOR ADDITION AND SUBTRACTION

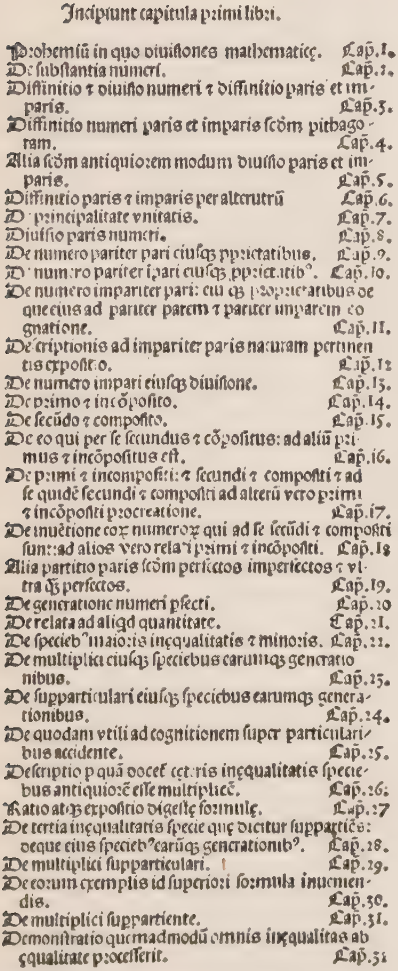
These symbols represent a pair of legs walking in the direction of the writing for addition, and reversed for subtraction. (From Rhind papyrus.)

Greek Arithmetic

Euclid's "Elements." The first great mathematical textbook of the Greeks is Euclid's Thirteen Books of the *Elements*, the Geometry written about 320 B.C. and continuing in active use almost to the present day. This work contains in books seven, eight, and nine a treatise on theoretical arithmetic, numbers being represented by

geometrical lines. In this work no explanation is given of the fundamental operations but rather properties of numbers now treated in number theory (*see* Chapter I). The proofs are by the rigid logical processes of the Greek geometry. Problems analogous to finding the greatest common divisor of two or more numbers and the least common multiple are treated incidentally, but not applied to fractions as in our arithmetic. The fraction idea is treated under proportion and found application in the theory of music, long regarded as a mathematical science.

Speculative or theoretical arithmetic. The Greek arithmetical treatise by Nicomachus, translated into Latin by Boethius, continued in active use well into the seventeenth century, and was used in European church schools almost exclusively for the subject of arithmetic in the tenth, eleventh, and twelfth centuries. Two distinct types of speculative arithmetic were current, the Boëthian and a mystical arithmetic involving contemplation of the numbers appearing in the Bible. Long after the invention of printing both types continued to flourish. The Boëthian type in diluted form, more verbose and even less mathematical than the original, is represented by the treatises of the other Romans, Martianus Capella (*c.* 410 A.D.) and Cassiodorus (*c.* 470–*c.* 564 A.D.). Slightly better is the *Arithmetica Speculativa* published in 1495, written by Thomas Bradwardine (*c.* 1290–1349), professor at Oxford and later Archbishop of Canterbury, and the *Tractatus proportionum* of Albert of Saxony (*c.* 1330), published in 1470. In many later textbooks of practical arithmetic the Boëthian number theory was given as introductory to the practical work.


Incipiunt capitula primi libri.

Prohemiũ in quo diuisiones mathematicę. Cap̃.1.
De substantia numeri. Cap̃.2.
Diffinitio ⁊ diuisio numeri ⁊ diffinitio paris et imparis. Cap̃.3.
Diffinitio numeri paris et imparis scd̃m pithagoram. Cap̃.4.
Alia scd̃m antiquiorem modum diuisio paris et imparis. Cap̃.5.
Diffinitio paris ⁊ imparis per alterutrũ Cap̃.6.
De principalitate vnitatis. Cap̃.7.
Diuisio paris numeri. Cap̃.8.
De numero pariter pari eiusq; proprietatibus. Cap̃.9.
De numero pariter ĩpari eiusq; proprietatib⁹. Cap̃.10.
De numero impariter pari: eiusq; proprietatibus de que eius ad pariter parem ⁊ pariter imparem cognatione. Cap̃.11.
Descriptionis ad impariter paris naturam pertinentis expositio. Cap̃.12
De numero impari eiusq; diuisione. Cap̃.13.
De primo ⁊ incõposito. Cap̃.14.
De secũdo ⁊ composito. Cap̃.15.
De eo qui per se secundus ⁊ cõpositus: ad aliũ primus ⁊ incõpositus est. Cap̃.16.
De primi ⁊ incompositi: ⁊ secundi ⁊ compositi ⁊ ad se quidẽ secundi ⁊ compositi ad alterũ vero primi ⁊ incõpositi procreatione. Cap̃.17.
De inuẽtione eorũ numerorũ qui ad se secũdi ⁊ compositi sunt: ad alios vero relati primi ⁊ incõpositi. Cap̃.18
Alia partitio paris scd̃m perfectos imperfectos ⁊ vltra q̃ perfectos. Cap̃.19.
De generatione numeri pfecti. Cap̃.20
De relata ad aliqd quantitate. Cap̃.21.
De specieb⁹ maioris inęqualitatis ⁊ minoris. Cap̃.22.
De multiplici eiusq; speciebus earumq; generationibus. Cap̃.23.
De superparticulari eiusq; speciebus earumq; generationibus. Cap̃.24.
De quodam vtili ad cognitionem super particularibus accidente. Cap̃.25.
Descriptio p quã docet cęteris inęqualitatis speciebus antiquiorẽ esse multiplicẽ. Cap̃.26.
Ratio atq; expositio digestę formulę. Cap̃.27
De tertia inęqualitatis specie quę dicitur superpartiẽs: deque eius specieb⁹ earũq; generationib⁹. Cap̃.28.
De multiplici superparticulari. Cap̃.29.
De eorum exemplis id superiori formula inueniendis. Cap̃.30.
De multiplici superpartiente. Cap̃.31.
Demonstratio quemadmodũ omnis inęqualitas ab ęqualitate processerit. Cap̃.32

The chapter headings indicate the speculative character of the material. Chapter eight takes up the division of even numbers (*Diuisio paris numeri*) followed by a discussion in chapters nine, ten, and eleven of "evenly-even" numbers, "evenly-odd," and of "oddly-even" numbers. "Evenly-even" are numbers of the form 2^n; that is, 32; when divided repeatedly by 2 these lead finally to unity.

THE ARITHMETIC OF BOETHIUS FROM HIS "OPERA OMNIA," PUBLISHED AT BASLE IN 1493

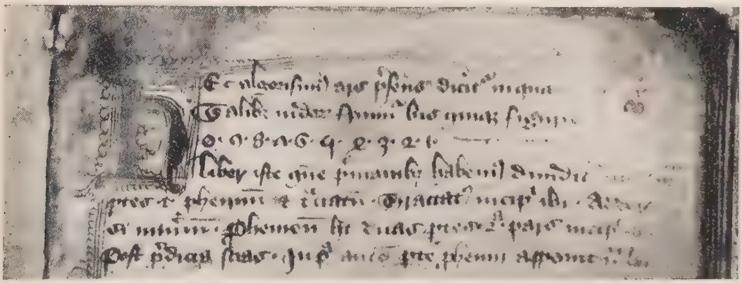
Mystical arithmetic. The second type of speculative arithmetic is well represented by the *Mysticae Numerorum* by Petrus Bongus of Bergamo, Italy, which appeared in 1583–84 and enjoyed seven editions. Some 400 pages are devoted to the discussion of the numbers from one to ten, in each case with copious references to Biblical material. A beginning was made along the same line about a thousand years earlier by Isidore of Seville, born in the year 570 A.D., who wrote further a short extract along Boëthian lines in his *Origines* or *Etymologies*.

Hindu and Arabic Texts

Hindu texts. The Hindu treatise on arithmetic by Aryabhata consists of 108 couplets of verse, intended to be memorized and requiring extensive interpretive explanation. Brahmagupta in the seventh century, Mahaviracarya in the ninth, and Bháskara in the eleventh wrote quite extensive systematic treatises including arithmetical and algebraical material and mensuration. The Arabs drew both method and content of arithmetic and algebra from Hindu sources, but the precise texts which they utilized are not known.

Arabic texts. The Arabs rendered a great service to the progress of civilization by writing admirable textbooks on arithmetic, algebra, and trigonometry, and on other subjects as well. Between 750 A.D. and 1450 A.D., some five hundred Arabs wrote treatises, whose fame at least has survived, on mathematical and astronomical subjects. So far as service to elementary mathematics is concerned the Persian Mohammed ibn Musa al-Khowarizmi, who lived at Bagdad in the ninth century, enjoys the greatest

distinction. His arithmetic in Latin translation, with his name *Algorismus* concealed as a title, continued in direct use until the fourteenth century, while his algebra was used in Latin translation until the sixteenth century. The Arabs had real ability in devising proper problems and methods for instruction. Extended lists of their algebraical and arithmetical problems appeared in the work of Leonard of Pisa in 1202, revised by him in 1228,



Trinity College MS. 0·1·31, Cambridge, England

THE "CARMEN DE ALGORISMO" OF ALEXANDRE DE VILLE DIEU

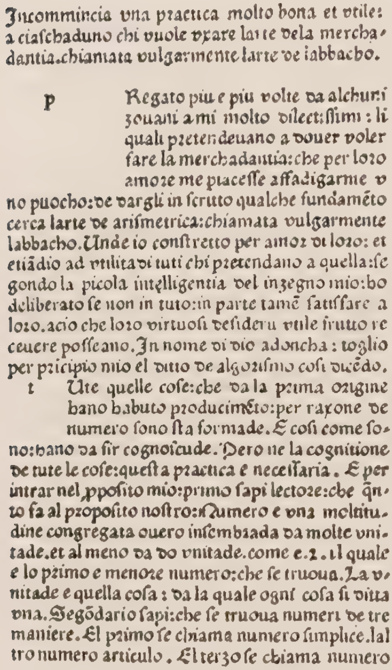
Verses from the *Carmen* of Alexandre de Ville Dieu: "Hec algorisimus ars praesens dicitur in qua . . . . 0· 9· 8· 7· 6· 5· 4· 3· 2· 1. The following section gives a fourteenth-century commentary.

and also in the *Summa d'Arithmetica* of 1494 by Luca Paciuolo, which includes the first printed algebra. For further centuries numerous other mathematical texts continued to present the Arabic material, frequently quite unaltered.

European Textbooks after 1200

Two popular texts. The two most popular European textbooks on the new arithmetic with the zero were undoubtedly the *Algorismus Vulgaris* of John of Halifax (Sacrobosco) and the *Carmen de Algorismo* of Alexandre

de Ville Dieu, both written during the first half of the thirteenth century. Until the invention of printing


Incommincia vna practica molto bona et vtile: a ciascbaduno chi vuole vxare larte dela merchadantia.chiamata vulgarmente larte de labbacbo.

p Regato piu e piu volte da alchuni zouani a mi molto dilectissimi: li quali pretendeuano a douer voler fare la merchadantia: che per loro amore me piacesse affadigarme vno puocho: de dargli in scritto qualche fundamēto cerca larte de arismetrica: chiamata vulgarmente labbacbo. Unde io constretto per amor di loro: et etiādio ad vtilita di tuti chi pretendano a quella: segondo la picola intelligentia del inzegno mio: bo deliberato se non in tuto: in parte tamē satisfare a loro. acio che loro virtuosi desideru vtile frutto receuere posseano. In nome di dio adoncha: toglio per pricipio mio el ditto de algorismo cosi dicēdo.

t Ute quelle cose: che da la prima origine bano babuto producimēto: per raxone de numero sono sta formade. E cosi come sono: bano da fir cognoscude. Pero ne la cognitione de tute le cose: questa practica e necessaria. E per intrar nel pposito mio: primo sapi lectore: che qnto fa al proposito nostro: Numero e vna moltitudine congregata ouero insembrada da molte vnitade. et al meno da do vnitade. come e. 2. il quale e lo primo e menore numero: che se truoua. La vnitade e quella cosa: da la quale ogni cosa si ditta vna. Segōdario sapi: che se truoua numeri de tre maniere. El primo se chiama numero simplice. lal tro numero articulo. El terzo se chiama numero

FIRST PRINTED ARITHMETIC

The Treviso Arithmetic of 1478. The first separate treatise on practical arithmetic was printed at Treviso in Italy in 1478. The author is not known.

The blank spaces are left for ornamental initials.

The author states that he writes the work at the earnest solicitation of numerous students.

these works held almost undisputed sway in university circles. Hundreds of copies were made by students, undoubtedly taken by dictation; the master would read two or three lines, the student copying, and then the master would discourse upon the meaning.

Arithmetics in print. The invention of printing made popular the longer commercial arithmetics which appeared first in Italy and Germany and then in other European

countries. Undoubtedly commercial activities stimulated both the appearance and the use of these textbooks.

RECORD'S
ARITHMETICK:
OR,
The Ground of Arts:
TEACHING
The perfect work and practise of Arithmetick, both in whole Numbers and Fractions, after a more easie and exact form then in former time hath been set forth:
Made by Mr. *Robert Record*, D. in Physick.

Afterward augmented by Mr. John Dee

And since enlarged with a Third part of *Rules of Practise*, abridged into a briefer method then hitherto hath been published, with divers necessary *Rules* incident to the *Trade of Merchandise*: with *Tables* of the Valuation of all *Coynes*, as they are currant at this present time.

By *John Mellis*.

And now diligently perused, Corrected, Illustrated and Enlarged; with an *Appendix* of figurative Numbers, and the extraction of their Roots, according to the method of *Christian Urstitius*: with Tables of Board and Timber measure; and new Tables of Interest, after 10. and 8. *per* 100. with the true value of Annuities to be bought or sold, present, Respited, or in Reversion: The first calculated by *R.C.* but corrected, And the latter diligently calculated by *Ro. Hartwell, Philomathemat.*

Scientia non habet inimicum nisi ignorantem.
Fide —— sed —— Vide

LONDON
Printed by *James Flesher*, and are to be sold by *Edward Dod* at the signe of the Gun in Ivie-lane. 1654.

TITLE PAGE OF RECORDE'S ARITHMETIC

The first edition of Recorde's popular arithmetic appeared about 1542.

Authors had a weakness for the type of title here shown. Recorde entitled his astronomical work of 1556 *The Castle of Knowledge;* his geometry was called *The Pathwaie to Knowledge;* his algebra was entitled *The Whetstone of Witte* (London, 1557).

Editions of the arithmetic continued to appear until 1700 A.D.

The first widely popular compendium of mathematics is the so-called *Summa d' Arithmetica* of Luke Paciuolo of 1494; the *Liber Abbaci* by Leonard of Pisa, written in 1202, is of similar nature, and there were several others written in Latin in the fourteenth and fifteenth centuries. In the encyclopedic works of the same period the treatment of mathematics is generally quite fragmentary, not sufficient to be counted as texts. The material on arithmetic in Caxton's *Mirrour of the World* (*see* page 58)

is an excellent illustration. The *Margarita Philosophica* by Gregorius Reisch (1501), a sixteenth-century encyclopedia often reprinted, contains an excellent treatise on arithmetic.

Between the invention of printing and 1500 A.D. there

Von verkerten fragen.

tag zu marckt kauffet einē Narten voll hering/ nemlich 180. darumb gibt er 18. alb. Ist die frag/ wie viel Hering hat er für 1. alb. Machs wie obstehet/ so kompt dir gerad 10. So vil Hering hat er für ein alb. kaufft.

Verkerte Frag.

So dir aber ein verkerte Frag fürkäme/ so solt du auch die zal derselben fragen/ auff die Regel de Tri verkert/ ordnen vnnd schreiben.

Zu eim Exempel.

Einer sprech/ er hett 6. eln Tuch vmb 24. alb. kaufft/ wie viel eln er kauffen möcht vmb 48. alb. In diser vñ dergleichen fragen/ must du die zwey zal/ die inn die mitte gehört/ vor setzen/ vnd also gedencken/ ordnen vnd schreiben/ 24. geben mir 6. was geben mir 48. Vnd machs als dann nach der Regel/ so erferestu/ daß 12. Eln 48. alb. (das ist ein gulden vnd 12. alb.) vnd ein Ele 4. alb. kost.

Hierauff

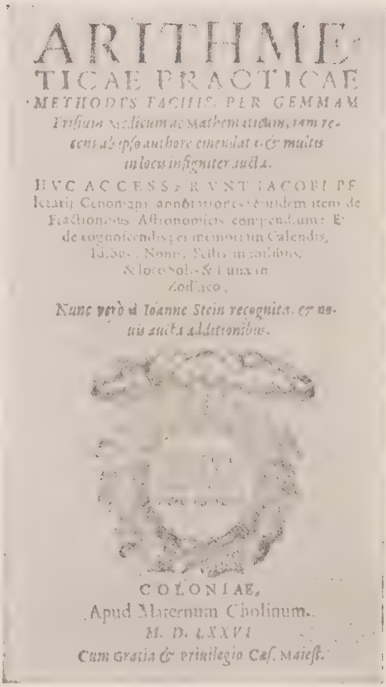
ILLUSTRATED PROBLEMS, "ON REVERSED QUESTIONS," FROM KÖBEL'S ARITHMETIC OF 1584

The first problem concerns a barrel of herring bought for 18 "alb," containing 180 herring. How many herring for 1 "alb"?

The second problem involves the Rule of Three, or "Regel de Tri."

appeared some thirty practical arithmetics of which more than one-half were in Latin, seven in Italian, four in German, and one in French. During this period there appeared about twenty-six editions in Latin of the theoretical arithmetics, along Boëthian lines. Up to 1514 the arithmetics in Latin greatly exceeded all others,

but in 1514 Köbel's *Rechenbiechlin* in German marks the opening of the era in which the mother tongue began to be used in instruction. There were over thirty sixteenth-century editions in German of Köbel's three books on arithmetic. The Italian commercial arithmetic of Borghi of 1484 passed through seventeen editions up to 1577, while the work of 1515 in Italian by Girolamo and Giannantonio Tagliente achieved thirty-six editions up to 1586. Adam Riese's arithmetics beginning in 1522 eclipsed all others in the vernacular, having more than forty sixteenth-century editions and several of the seventeenth century. Strangely enough, the arithmetic to enjoy the greatest number of editions in the sixteenth century, over sixty, was a Latin treatise by Gemma Rainer Frisius (1508–1555), a Dutch physician; this *Arithmeticae practicae methodus facilis* appeared first in 1540, with some later editions modified by a Frenchman, Jacques Peletier of Mans, and other editions modified further by the German, Jacob Stein.



ARITHMETICAE PRACTICAE
METHODVS FACILIS, PER GEMMAM
Frisium Medicum ac Mathematicum, iam recens ab ipso authore emendata, & multis in locis insigniter aucta.

HVC ACCESSERVNT IACOBI PELetarij Cenomani annotationes: eiusdem item de Fractionibus Astronomicis compendium: Et de cognoscendis per memoriam Calendis, Idibus, Nonis, Festis mobilibus, & loco Solis & Lunae in Zodiaco.

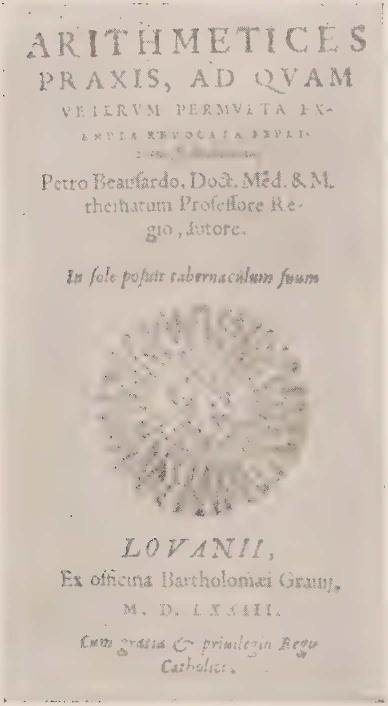
Nunc verò à Ioanne Stein recognita, & nouis aucta additionibus.

COLONIAE,
Apud Maternum Cholinum.
M. D. LXXVI
Cum Gratia & privilegio Caes. Maiest.

TITLE PAGE

This arithmetic by Gemma Frisius was the most popular treatise in Latin during the sixteenth century.

Fifteenth-century arithmetics. The first printed arithmetic is an anonymous work of the commercial type which appeared in Treviso, Italy, in 1478, written in Italian.


ARITHMETICES
PRAXIS, AD QVAM

Petro Beausardo, Doct. Med. & Mathematum Professore Regio, autore.

In sole posuit tabernaculum suum

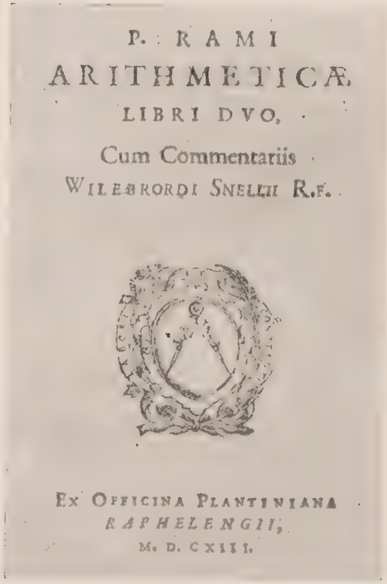
LOVANII,
Ex officina Bartholomæi Grauij,
M. D. LXXIII.
Cum gratia & priuilegio Regis Catholici.

TITLE PAGE

This arithmetic appeared at Louvain, one of the centers of mathematical learning during the sixteenth century.

The first printed Latin treatise on the new numerals was written by Prosdocimo de Beldamandi (died 1428), a professor at the University of Padua, Italy. This was printed in 1483 at Padua with the title: *Prosdocimi de beldamandis algorismi tractatus perutilis*, etc. The work includes the first treatise in print on fractions, by Johannes de Liveriis, a Sicilian astronomer (*c.* 1300–1350); here is found the designation "de minutijs tam vulgaribus quam physicis," having reference to vulgar fractions and astronomical minutes and seconds. Preceding this work by one year comes a German treatise, printed at Bamberg; the author was Ulrich Wagner, a Nürnberg *Rechenmeister*, or professional teacher of reckoning. The first illustrated arithmetic appears to be by Johann Widman of Eger, a German commercial work published at Pforzheim in 1489.

Sixteenth-century arithmetics. About nine hundred arithmetics appeared during the sixteenth century; they are listed and described in David Eugene Smith's *Rara Arithmetica*, which is a descriptive catalogue copiously illustrated of the great collection of arithmetics gathered by Mr. George A. Plimpton of New York.[1] Of the nine hundred arithmetics about four hundred are in Latin, and of those one-quarter are of the Boëthian type of number theory, or on mysticism of numbers. About two hundred commercial arithmetics appeared in German, followed closely in number and content by Italian arithmetics. The German writers date largely from 1550 to 1600, whereas the Italian writers are largely of the earlier period. Similarly the English (45), French (40), Spanish (25), and Dutch (15) arithmetics are almost exclusively from the second



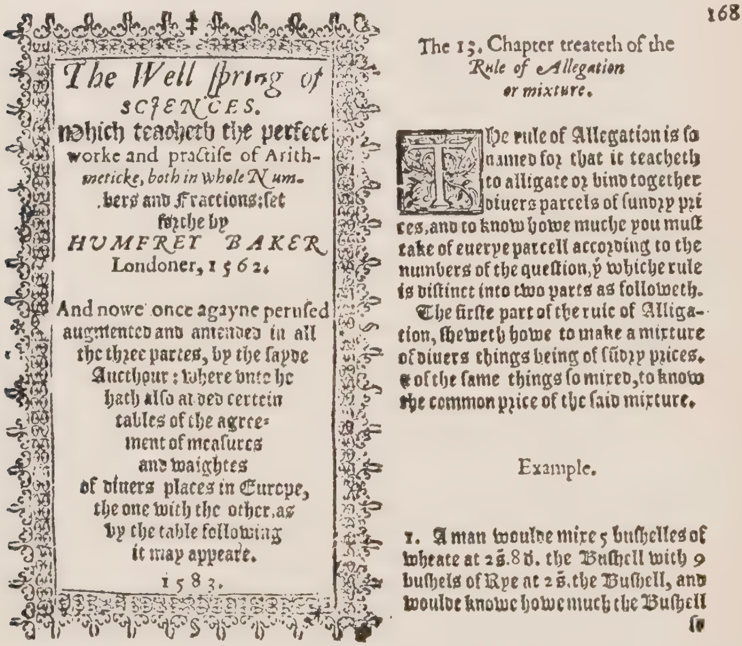
TITLE PAGE

The widely popular arithmetic of Peter Ramus, with commentary by Snell, who first stated the law of refraction.

The work is the product of the famous Plantin Press; the compasses constitute a part of the printer's device or trademark.

[1] Mr. Plimpton with great generosity has placed this wonderful library at the service of many workers in the field of the history of science; the writer is deeply indebted to Mr. Plimpton for many courtesies over a long period of years.

half of the sixteenth century. During this latter period the production of arithmetics in Latin dropped to less than one-quarter of the total number as opposed to one-half the total during the preceding part of the century.



The Well ſpring of
ſciences.
which teacheth the perfect
worke and practiſe of Arith-
meticke, both in whole Num-
bers and Fractions: ſet
forthe by
HUMFREY BAKER
Londoner, 1562.

And nowe once agayne peruſed
augmented and amended in all
the three partes, by the ſayde
Aucthour: where unto he
hath alſo added certein
tables of the agree-
ment of meaſures
and waightes
of diuers places in Europe,
the one with the other, as
by the table followinge
it may appeare.
1583.

168

The 15. Chapter treateth of the
*Rule of Allegation
or mixture.*

The rule of Allegation is ſo
named for that it teacheth
to alligate or binde together
diuers parcels of ſundry pri-
ces, and to know howe muche you muſt
take of euerye parcell according to the
numbers of the queſtion, ẏ whiche rule
is diſtinct into two parts as followeth.
The firſte part of the rule of Alliga-
tion, ſheweth howe to make a mixture
of diuers things being of ſūdry prices,
& of the ſame things ſo mixed, to know
the common price of the ſaid mixture.

Example.

1. A man woulde mixe 5 buſhelles of
wheate at 2 s. 8 d. the Buſhell with 9
buſhels of Rye at 2 s. the Buſhell, and
woulde knowe howe much the Buſhell
ſo

TWO PAGES FROM BAKER'S ARITHMETIC

Baker's arithmetic was second in popularity to Recorde's treatise. Both works were largely based on Italian commercial arithmetics.

In the seventeenth century the arithmetics in Latin diminished in use and in number published. The total number of arithmetics published during the seventeenth century, including editions, would doubtless approximate two thousand.

Seventeenth and eighteenth centuries. Popular interest in arithmetic and general instruction in the subject increased so rapidly after the sixteenth century that hundreds of books appeared to supply the new demand. The continental texts were, on the whole, more scientific than those employed in England. The encyclopedic works touching all subjects remotely connected with mathematics were common. In England one popular compendium for self-instruction included arithmetic with reading, writing, bookkeeping, with instructions for carpenters, bricklayers, and the like, and with dialing, pickling, and a list of fairs in England. The English texts directly fashioned the American arithmetic; in fact, the majority of texts used in America up to 1810 were either imported from England or were American reprints of English works.

ARITMETICA
PRATTICA
COMPOSTA DAL MOLTO
Reueren. Padre Chriſtoforo Clauio
Bambergenſe nella Compagnia di GIESV.

Et tradotta da Latino in Italiano dal S. Lorenzo Caſtellano Patritio Romano.

Reuiſta dal medeſimo Padre Clauio
con alcune aggiunte.

IN ROMA,
Per Guglielmo Facciotti. M. DC. XIII.

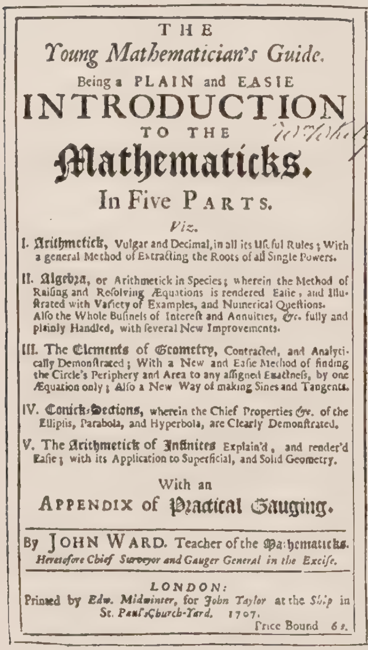
CON LICENZA DE' SVPERIORI.

TITLE PAGE

The Jesuit, Christopher Clavius, wrote a whole series of treatises on elementary mathematics and astronomy. This work appeared first in Latin at Rome in 1583, and enjoyed numerous editions.

Recorde and Baker continued in use during the seventeenth century, being gradually supplanted by treatises

devoting more attention to decimal fractions and to new commercial problems, and less attention to counters. Among the early successful texts of this class may be placed that with the title: *Mr. Blundevil: His Exercises*



THE
Young Mathematician's Guide.
Being a PLAIN and EASIE
INTRODUCTION
TO THE
Mathematicks.
In Five PARTS.
Viz.

I. Arithmetick, Vulgar and Decimal, in all its Useful Rules; With a general Method of Extracting the Roots of all Single Powers.

II. Algebra, or Arithmetick in Species; wherein the Method of Raising and Resolving Æquations is rendered Easie, and Illustrated with Variety of Examples, and Numerical Questions. Also the Whole Business of Interest and Annuities, &c. fully and plainly Handled, with several New Improvements.

III. The Elements of Geometry, Contracted, and Analytically Demonstrated; With a New and Easie Method of finding the Circle's Periphery and Area to any assigned Exactness, by one Æquation only; Also a New Way of making Sines and Tangents.

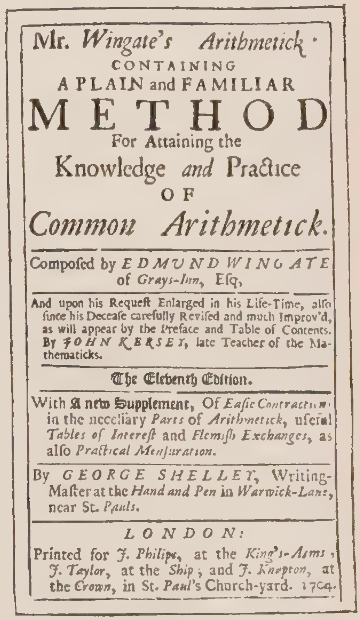
IV. Conick-Sections, wherein the Chief Properties &c. of the Ellipsis, Parabola, and Hyperbola, are Clearly Demonstrated.

V. The Arithmetick of Infinites Explain'd, and render'd Easie; with its Application to Superficial, and Solid Geometry.

With an
APPENDIX of Practical Gauging.

By JOHN WARD. Teacher of the Mathematicks.
Heretofore Chief Surveyor and Gauger General in the Excise.

LONDON:
Printed by *Edw. Midwinter*, for *John Taylor* at the *Ship* in St. *Paul's Church-Yard.* 1707.
Price Bound 6 *s.*



Mr. *Wingate's Arithmetick*:
CONTAINING
A PLAIN and FAMILIAR
METHOD
For Attaining the
Knowledge *and* Practice
OF
Common Arithmetick.

Composed by *EDMUND WINGATE* of *Grays-Inn*, Esq;

And upon his Request Enlarged in his Life-Time, also since his Decease carefully Revised and much Improv'd, as will appear by the Preface and Table of Contents. By *JOHN KERSEY*, late Teacher of the Mathematicks.

The Eleventh Edition.

With A new Supplement, Of *Easie Contractions* in the necessary *Parts* of *Arithmetick*, useful *Tables of Interest* and *Flemish Exchanges*, as also *Practical Mensuration.*

By *GEORGE SHELLEY*, Writing-Master at the *Hand and Pen* in *Warwick-Lane*, near St. *Pauls.*

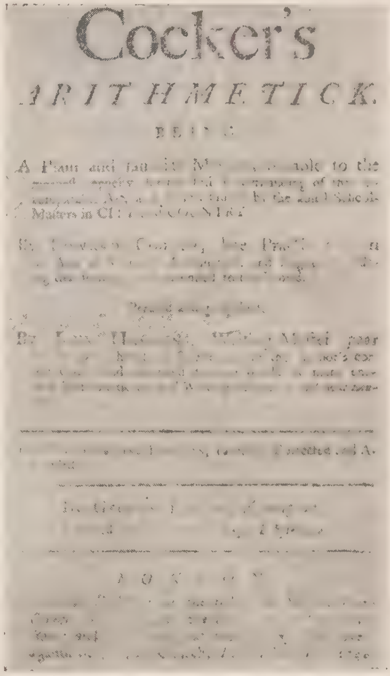
LONDON:
Printed for *J. Philips*, at the *King's-Arms*, *J. Taylor*, at the *Ship*; and *J. Knapton*, at the *Crown*, in St. *Paul's* Church-yard. 1704.

TWO TITLE PAGES

Two works widely popular in England and imported in large quantities by New England booksellers. In Harvard University Ward's treatise was used for a time as a textbook.

contayning Eight Treatises, and Edmund Wingate's *Arithmetique made Easie.* Of the former treatise the seventh edition of 1636 was "corrected and somewhat enlarged by R. Hartwell, Philomathematicus." Wingate's work appeared about 1629, and the edition of 1650 was a popular

revision by John Kersey. Probably the most significant addition by Kersey was the introduction of decimal


Cocker's
ARITHMETICK.

TITLE PAGE

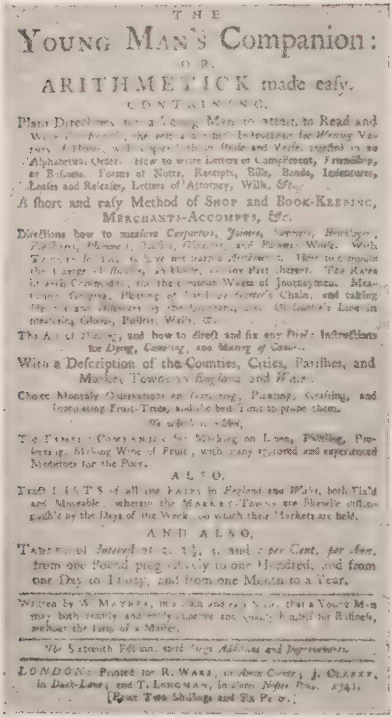
No arithmetic in the English language has had as many editions as Cocker, and only Recorde was used over as long a period of time. "George Fisher" is a pseudonym for one Mrs. Slack, concerning whom we know only that she wrote three or four treatises in the nature of compendiums which were among the most widely read books for instruction of the eighteenth century, both in England and in America.

The expressions, "Printed . . . at the *Bible* and *Sun* in *Amen-Corner*," . . . "at the *Red-Lion*," . . . "at the *Looking-Glass*," refer to signboards placed outside of the book shops similar to signs employed then by taverns.

fractions. Practical problems on tare, trett loss, gain, and barter were also added. Wingate continued to appear until late in the eighteenth century.

The most popular arithmetic, at least in number of editions, appears to have been Edward Cocker's *Arithmetick*, "Perused and published by J. Hawkins" in 1678, three years after the death of Cocker. Approximately one hundred editions were published in the British Isles, but strange to say only one edition (at most) appeared in

America. Cocker dropped the old terminology, condensed the presentation of most topics, particularly exchange,


THE

YOUNG MAN'S Companion:

OR,

ARITHMETICK made eafy.

CONTAINING,

Plain Directions [illegible] to Read and [illegible] Alphabetical Order. How to write Letters of Compliment, Friendship, or Bufinefs. Forms of Notes, Receipts, Bills, Bonds, Indentures, Leafes and Releafes, Letters of Attorney, Wills, &c.

A fhort and eafy Method of SHOP and BOOK-KEEPING, MERCHANTS-ACCOMPTS, &c.

Directions how to meafure Carpenters, Joiners, [illegible] Work. [illegible] the Common Wages of Journeymen. [illegible] Chain, and taking [illegible] Line in [illegible] Walls, &c.

[illegible] and how to direct and fix any [illegible] Inftructions for Dying, [illegible]

With a Defcription of the Counties, Cities, Parifhes, and Market Towns in England and Wales.

[illegible] Fruit-Trees, and the beft Time to prune them.

[illegible]

[illegible] Making Wine of Fruit, with many approved and experienced Medicines for the Poor.

ALSO,

[illegible] LISTS of all the FAIRS in England and Wales, both Fix'd and Moveable, wherein the MARKET-Towns are likewife diftinguifh'd by the Days of the Week on which their Markets are held.

AND ALSO,

TABLES of Intereft at [illegible] per Cent. per Ann. from one Pound [illegible] to one Hundred, and from one Day to [illegible], and from one Month to a Year.

Written by W. MATHER, [illegible] that a Young Man may both readily and [illegible] for Bufinefs, without the Help of a Mafter.

The Sixteenth Edition, with large Additions and Improvements.

LONDON: Printed for R. WARE, in Amen Corner; J. CLARKE, in Duck-Lane; and T. LONGMAN, in Pater-Nofter Row. [illegible]

[Price Two Shillings and Six Pence.]

TITLE PAGE

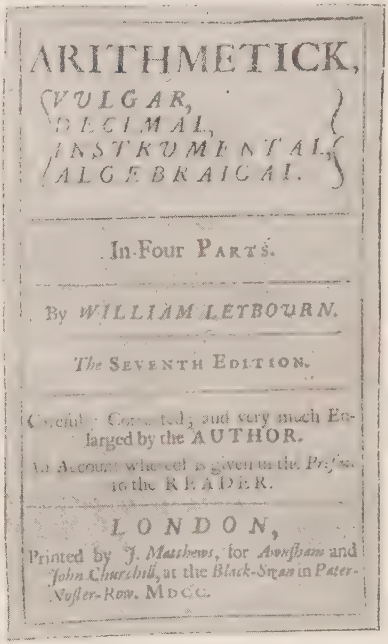
The type of textbook represented by Mather's *Young Man's Companion* was responsible in some measure for the introduction into arithmetic of problems on plastering, carpeting, masonry, and allied topics.

This work, like Fisher's *Instructor* and Bradford's *The Secretary's Guide*, was designed for self-instruction.

dropped the work on counters, and introduced many lists of problems.

Another popular book which appeared in the seventeenth century and continued in wide use during the eighteenth century was William Mather's *The Young Man's Companion, or Arithmetick Made Easy.* Leybourn's treatise on *Arithmetick, vulgar, decimal, instrumental, algebraical*, was more scientific but not so widely

used. Both texts were used by William Bradford in the preparation of the first American arithmetic in English.



ARITHMETICK,

VULGAR,
DECIMAL,
INSTRUMENTAL,
ALGEBRAICAL.

In Four PARTS.

By WILLIAM LEYBOURN.

The SEVENTH EDITION.

Carefully Corrected; and very much Enlarged by the AUTHOR.

An Account whereof is given in the Preface to the READER.

LONDON,

Printed by J. Matthews, for Awnsham and John Churchill, at the Black-Swan in Pater-Noster-Row. MDCC.

TITLE PAGE

Leybourn was a well-known teacher and surveyor.

Leybourn s numerous treatises included one on recreations, long a favorite topic with mathematicians. Among the problems given in his *Pleasure with Profit* (London, 1693) is that "to put five odd numbers together to make 20," which Leybourn designates as a *Falacy;* the answer is, "Three nines turned upside down, and two units." Augustus De Morgan thinks "the question more than answered, viz., in *very* odd numbers."

Among the earliest Ready-Reckoners was the one by Leybourn entitled: *Pan-Arithmologia; Being a Mirror for Merchants, a Breviate for Bankers, a Treasure for Tradesmen, a Mate for Mechanics,* . . . (London, 1694).

In his decimal arithmetic Leybourn expanded the treatment of interest, which was well begun by Mellis about 1630 in revisions of Recorde's *Grounde of Artes.* Leybourn took up discount, rebate, and equation of payments; bricklaying and similar topics were also treated by him as a part of arithmetic. Even more scholarly than Leybourn was William Oughtred, inventor of the slide rule, whose work, *The Key of Mathematicks New Forged and Filed,* appeared in London in 1647. Oughtred gave more attention to algebra, treated by him first in his Latin

Arithmeticae . . . institutio of 1631; the Latin version was more often reprinted than the English.

In the eighteenth century George Fisher's *Instructor* and Dilworth's *The Schoolmaster's Assistant* were about equally popular in England with Cocker; in America, from 1750 to 1800, these two were by far the most popular treatises in use.

American Works

Mexican arithmetics. The first arithmetical work printed in America appeared in Mexico, in 1556, the *Sumario compendioso* of Juan Diez Freyle. This work, primarily concerning the valuation of silver and gold, contains about twenty-five pages on arithmetic and algebra. Juan Bellevedcr published a similar work in Lima, Peru, in 1597, but no copy has been located; whether it contains any arithmetical material is problematical. In 1623 Pedro Paz published in Mexico the *Arte menor de Arithmetica Practica.* This is quite certainly the first arithmetic of America. The second was a treatise, *Arte de Arismetrica* by Don A. Reaton, probably also in Spanish, published in Mexico in 1649. The writer has not been able to locate a copy of either work. These arithmetics precede by about one hundred years any similar works in what is now the United States. The first university in America was founded in 1554 in Mexico, and the first lecturer there on mathematics, beginning his work in the latter part of the sixteenth century, was Juan Negrete, of whom little else is known.

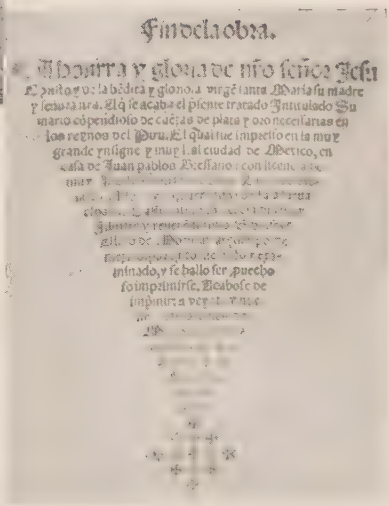
Early colonial arithmetics. In the American colonies the English textbooks on arithmetic which were imported

Sumario cōpēdioso delas quētas de plata y oro q̄ en los reynos del Piru son necessarias a los mercaderes: y todo genero de tratantes. Cō algunas reglas tocantes al Arithmetica.

Fecho por Juan Diez freyle.

Title page of the first work touching arithmetic to appear on the western continent.

The *Sumario compendioso de las quentas de plata y oro* is concerned primarily with the valuation of silver and gold of different degrees of refinement. The author interjected a discussion of arithmetic pertinent to the text and further a more academic discussion of algebra.



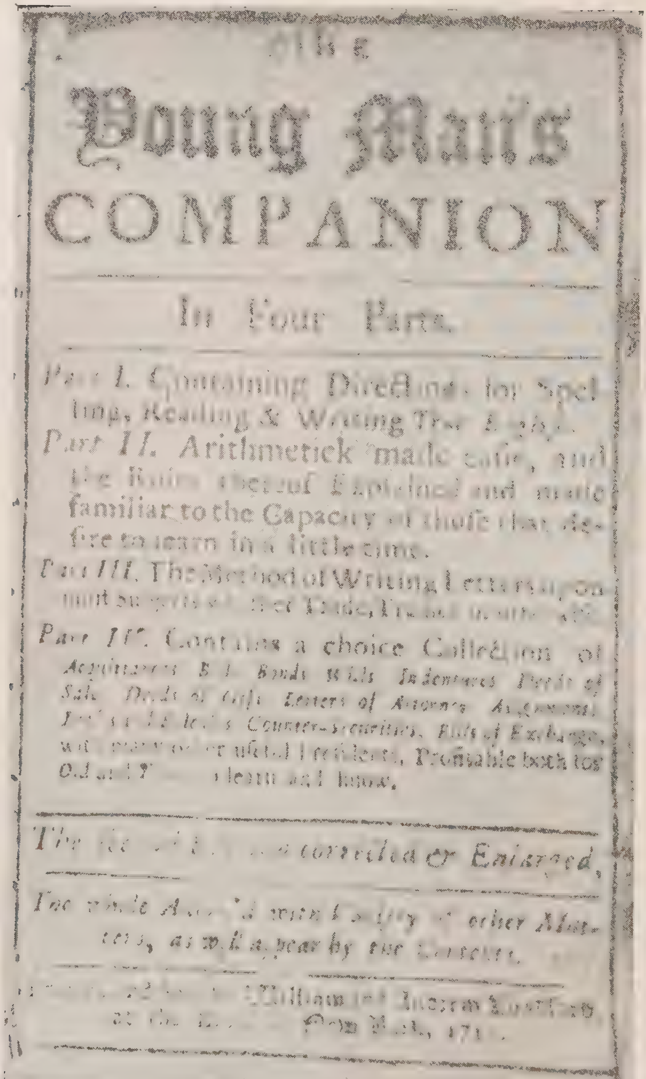
Fin dela obra.

A honrra y gloria de nr̃o señor Jesu Christo y dela bēdita y gloriosa virgē sancta Maria su madre y señora nr̃a. Aq̄ se acaba el pñte tratado Intitulado Sumario cōpendioso de cuētas de plata y oro necessarias en los reynos del Piru. El qual fue impresso en la muy grande ynsigne y muy leal ciudad de Mexico, en casa de Juan pablos Bressano: con licencia de

... minado, y se hallo ser prouecho so imprimirse. Acabose de imprimir a ...

This colophon states that the work was printed in Mexico by Juan Pablos, the first American printer. Mention is made of the permission to print given by Don Luis de Velasco, Viceroy of New Spain, and the further necessary permit to print given by the Archbishop of Mexico, Don Alonso de Montufar.

TITLE PAGE AND COLOPHON OF FREYLE'S MEXICAN WORK (1556)


THE

Young Man's

COMPANION

In Four Parts.

Part I. Containing Directions for Spelling, Reading & Writing True English.

Part II. Arithmetick made easie, and the Rules thereof Explained and made familiar to the Capacity of those that desire to learn in a little time.

Part III. The Method of Writing Letters [illegible] Trade, [illegible]

Part IV. Contains a choice Collection of *Acquittances,* [illegible] *Bonds, Wills, Indentures, Deeds of Sale, Deeds of Gift, Letters of Attorney, Assignments,* [illegible] *Counter-securities, Bills of Exchange,* with many other useful [illegible], Profitable both for *Old* and *Y*[illegible] learn and know.

The [illegible] *corrected & Enlarged,*

The whole [illegible] *with* [illegible] *other Matters, as will appear by the Contents.*

[illegible] William and Andrew Bradford, at the [illegible] New-York, 171[illegible]

FIRST KNOWN AMERICAN ARITHMETIC IN ENGLISH
The earlier edition of 1705 has entirely disappeared.

were largely Hodder's, Cocker's, and Ward's *Arithmeticks* followed closely by those of Wingate and Johnson with occasionally a Recorde. Works on geometry seem to have been imported rarely, whereas Norwood's *Trigonometry* and his *Navigation*, Seller's *Navigation*, and works on surveying were relatively common. In Harvard and Yale and other early universities Latin treatises on astronomy and mathematics were used in the seventeenth century, being gradually replaced by English texts in the eighteenth century.

In the United States the first extended treatment of arithmetic was the work of the dominating personality in the early history of printing in New York, William Bradford. His treatise, *The Secretary's Guide*, appeared in 1705, and enjoyed at least seven editions, 1705, 1710, 1719, 1728, 1729, 1737, and 1738. In Part II, we have: "Arithmetick made easie," which mirrors in the past a present American tendency. A similar work by one George Fisher, *The American Instructor; or, Young Man's Best Companion*, was based on an English text; it appeared first in America at Philadelphia in 1748 and in a dozen editions before 1800. The first arithmetic as a separate treatise printed in the United States was Hodder's (*see* illustrations, page 82), published at Boston in 1719. Isaac Greenwood, Professor of Mathematics at Harvard from 1727 to 1738, published the first arithmetic (*see* pages 82, 117) by an American in 1729. The next printed work on the subject appeared in New York in 1730 in Dutch (*see* page 83), containing also the first algebraic work in print in what is now the United States.

In the second half of the seventeenth century children were taught to "cypher" and frequently "to cast accounts" in many of the free schools of New England and New Amsterdam. Nevertheless arithmetic was not required for college entrance until the middle of the

HODDER's

ARITHMETICK:

OR, THAT

Neceſſary ART

Made moſt Eaſy.

Being explained in a way familiar to the Capacity of any that deſire to learn it in a little Time.

By JAMES HODDER, Writing-Maſter.

The Five and Twentieth Edition, Reviſed, Augmented, and above a Thouſand Faults Amended, by HENRY MOSE, *late Servant and Succeſſor to the Author.*

BOSTON: Printed by *J. Franklin*, For *S. Phillips, N. Buttolph, B. Elliot, D. Henchman, G. Phillips, J. Elliot*, and *E. Negus*, Bookſellers in *Boſton*, and Sold at their Shops. 1719.

ARITHMETICK

Vulgar and *Decimal*;

WITH THE

APPLICATION

THEREOF, TO

A VARIETY of CASES

IN

Trade, and *Commerce*.

BOSTON: *N. E.*

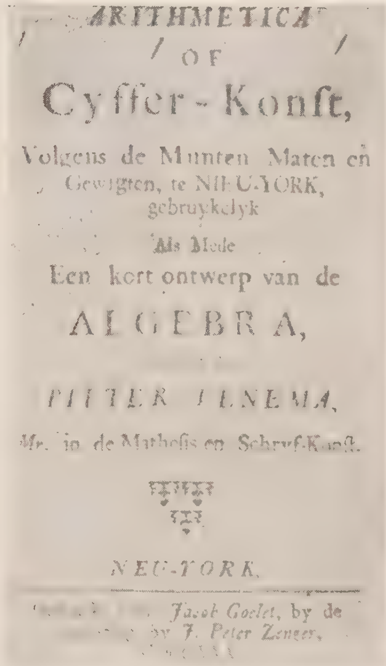
Printed by S. KNEELAND and T. GREEN, for T. HANCOCK at the Sign of the Bible and Three Crowns in *Annſtreet*. MDCCXXIX.

THE FIRST SEPARATE TREATISES TO APPEAR IN COLONIAL AMERICA

That on the right is the first separate treatise on arithmetic by a native; the work of Isaac Greenwood.

eighteenth century. John Burnham's *Arithmetick for the use of farmers and country people*, New London, 1748, is worthy of mention, although no copy has been preserved. Equally worthy of note is David Kendall's *The Young Lady's Arithmetic* of 1797. The popular arithmetics in the eighteenth century were Fisher's

American Instructor, Dilworth's *Schoolmaster's Assistant*, and Nicholas Pike's arithmetic of 1788, *A New and Complete System of Arithmetic*, which went through many editions in complete and in abridged form. Pike's arithmetic was the recognized American arithmetic from 1788 well into the nineteenth century.


ARITHMETICA
OF
Cyffer-Konst,
Volgens de Munten Maten en Gewigten, te NIEU-YORK, gebruykelyk
Als Mede
Een kort ontwerp van de
ALGEBRA,
PIETER VENEMA,
Mr. in de Mathesis en Schryf-Konst.
NEU-YORK,
Jacob Goelet, by de
by J. Peter Zenger,

TITLE PAGE

For many years after the English occupation the Dutch in New York continued the use, in instruction, of the mother tongue.

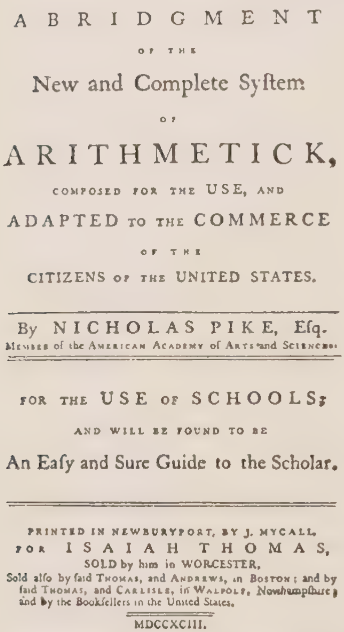
The author of this arithmetic began in Holland his activity in writing mathematical treatises.

The printer, J. Peter Zenger, is famous in the history of the liberty of the press in America in connection with a lawsuit about articles which appeared in a newspaper which he published,

Thousands in colonial times learned the numerals from the *New England Primer*, which contained simply a list from one to one hundred of Roman and Hindu-Arabic forms.

History reflected in arithmetic. American textbooks of the Revolutionary War period, and equally those of

the Civil War period, reflect the conditions of the times. Teachers of arithmetic do well to use such arithmetics occasionally for illustrative material. This not only quickens an interest in American history but indicates

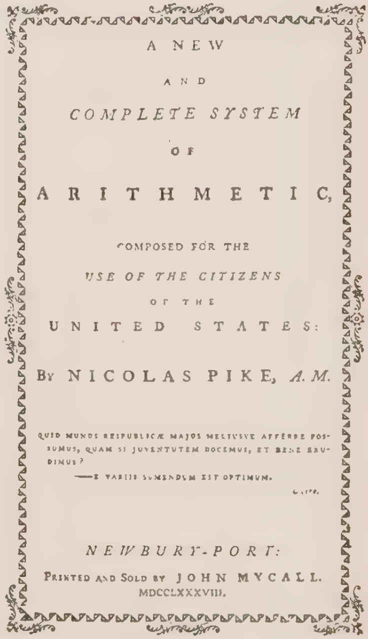

ABRIDGMENT
OF THE
New and Complete Syſtem
OF
ARITHMETICK,
COMPOSED FOR THE USE, AND
ADAPTED TO THE COMMERCE
OF THE
CITIZENS OF THE UNITED STATES.

By NICHOLAS PIKE, Eſq.
MEMBER of the AMERICAN ACADEMY of ARTS and SCIENCES.

FOR THE USE OF SCHOOLS;
AND WILL BE FOUND TO BE
An Eaſy and Sure Guide to the Scholar.

PRINTED IN NEWBURYPORT, BY J. MYCALL,
FOR ISAIAH THOMAS,
SOLD by him in WORCESTER,
Sold alſo by ſaid THOMAS, and ANDREWS, in BOSTON; and by ſaid THOMAS, and CARLISLE, in WALPOLE, Newhampſhire; and by the Bookſellers in the United States.

MDCCXCIII.


A NEW
AND
COMPLETE SYSTEM
OF
ARITHMETIC,
COMPOSED FOR THE
USE OF THE CITIZENS
OF THE
UNITED STATES:

By NICOLAS PIKE, *A.M.*

QUID MUNUS REIPUBLICÆ MAJUS MELIUSVE AFFERRE POSSUMUS, QUAM SI JUVENTUTEM DOCEMUS, ET BENE ERUDIMUS?
—E VARIIS SUMENDUM EST OPTIMUM.
CICERO.

NEWBURY-PORT:
PRINTED AND SOLD BY JOHN MYCALL.
MDCCLXXXVIII.

THE FIRST POPULAR AMERICAN ARITHMETICS APPEARING AFTER THE REVOLUTION

Pike continued in use until the middle of the nineteenth century.

the fact that arithmetic and mathematics are developments intimately connected with the whole history of culture.

How much of the history of the New World is reflected in the story of its arithmetic? The Spanish discoverers

are represented by the early Mexican works in Spanish; the English colonizers are represented by Hodder and the Dutch by Peter Venema; the French are represented by a Canadian arithmetic of 1809 in French; the native sons appear with William Bradford in 1705 and Isaac Greenwood in 1729, and come to a dominating place with Pike in 1788. Other great historical movements of American history are reflected in the arithmetics, worthy of somewhat attentive study by teachers who seek through arithmetic and other studies to educate the American youth.

List of Arithmetics and Arithmetical Works Published in America before 1800

1556 Juan Diez Freyle, *Sumario Compendioso de las quentas de plata y oro . . . . Con algunas reglas tocantes al Arithmetica.* Spanish. Mexico. Printed by Juan Pablos of Brescia. Copies in British Museum and in the Escorial. Photographic copy in the University of Michigan Library and library of David Eugene Smith, Columbia University.

1623 Pedro Paz, *Arte menor aprender todo el menor del Arithmetica, sin Maestro.* Mexico. Printed by Joan Ruyz. 2 l.+181 numbered folios+3 l. of tables; 21 chapters.

1649 Atanasius Reaton (Pasamonte), *Arte menor de Arismetrica.* Printed by Viuda de B. Calderon. 3 l.+78 numbered folios; 14 chapters.

1675 Benito Fernandez de Belo, *Breve arithmetica por el mas sucinto modo, que hasta oy se ha visto.* Mexico. Printed by Viuda de B. Calderon. John Carter Brown Library.

1705 William Bradford, *The Young Man's Companion.* New York. No copy known.

1710 William Bradford, *The Young Man's Companion.* In four parts. Part II. Arithmetick made easie, and the rules thereof Explained and made familiar to the Capacity of those that desire to learn in a little time. . . . . Printed

and Sold by William and Andrew Bradford, at the Bible in New York, 1710. Two imperfect copies in private hands; photographic copy in New York Public Library.

1719 WILLIAM BRADFORD, *The Secretary's Guide, or Young Man's Companion.*

1719 HODDER, Boston. Reprint of an English text. Printer: J. Franklin. 2 l., viii+216 pp. L. C.; A. A.

1728 WILLIAM BRADFORD, *The Secretary's Guide.* Part II, Arithmetick made easie. New York. W. Bradford. Pp. (2), (2), (6), 192. L. C.

1729 ISAAC GREENWOOD, *Arithmetick, Vulgar and Decimal.* Boston. Printers: S. Kneeland and T. Green. First separate text by a native of colonial America. Title, 158 pp., 4 pp. Index, and 4 pp. Adv. L. C.

WILLIAM BRADFORD, *The Secretary's Guide.* New York. Printer: Wm. Bradford. Pp. (2), (2), (6), 192.

1730 PETER VENEMA, *Arithmetica of Cyffer Konst.* Dutch, New York. Printer: J. Peter Zenger. 120 pp. N. Y. H. S.

1737 WM. BRADFORD, *The Secretary's Guide.* New York. Printer: Wm. Bradford. Pp. (2), (8), 248. N. Y. P.

1738 WM. BRADFORD, *The Secretary's Guide.* Philadelphia. Printer: Andrew Bradford. Pp. (2), (8), 248. P.

1748 JONATHAN BURNHAM, *Arithmetick for the use of farmers and Country people.* New London. Printer: T. Green.

1748 GEORGE FISHER, *The American Instructor; or, Young Man's Best Companion*, containing Spelling, Reading, Writing, and Arithmetick, in an easier Way than any yet published. Reprint of an English work. Philadelphia. Printer: Benjamin Franklin and D. Hall. Pp. v, 378; 5 plates. N. Y. P., P.

1749 *Same.* Boston.

1753 FISHER, *Young Man's Best Companion.* Philadelphia. Printer: Benj. Franklin and D. Hall. Pp. v, 384 (2); 6 plates. Hist. Soc. Penn.

1758 *Conclusiones Mathematicas . . . . por* DON FERNANDO DE

The American

INSTRUCTOR:

OR,

Young Man's Best Companion.

CONTAINING,

Spelling, Reading, Writing, and Arithmetick, in an easier Way than any yet published; and how to qualify any Person for Business, without the Help of a Master.

Instructions to write Variety of Hands, with Copies both in Prose and Verse. How to write Letters on Business or Friendship. Forms of Indentures, Bonds, Bills of Sale, Receipts, Wills, Leases, Releases, *&c.*

Also Merchants Accompts, and a short and easy Method of Shop and Book-keeping; with a Description of the several *American* Colonies.

Together with the *Carpenter*'s Plain and Exact Rule: Shewing how to measure *Carpenters, Joyners, Sawyers, Bricklayers, Plaisterers, Plumbers, Masons, Glasiers,* and *Painters* Work. How to undertake each Work, and at what Price; the Rates of each Commodity, and the common Wages of Journeymen, with *Gunter's* Line, and *Coggeshal's* Description of the *Sliding-Rule.*

Likewise the PRACTICAL GAUGER made Easy; the Art of *Dialling*, and how to erect and fix any *Dial*; with Instructions for *Dying, Colouring,* and making *Colours.*

To which is added,

The POOR PLANTERS PHYSICIAN.

With Instructions for *Marking* on *Linnen*; how to *Pickle* and *Preserve*; to make divers Sorts of Wine; and many *excellent Plaisters,* and *Medicines,* necessary in all *Families.*

And also

Prudent Advice to young *Tradesmen* and *Dealers.*

The whole better adapted to these American *Colonies, than any other Book of the like Kind.*

By *GEORGE FISHER*, Accomptant.

The Ninth Edition Revised and Corrected

PHILADELPHIA: Printed by B. FRANKLIN and D. HALL, at the New-Printing-Office, in *Market-Street*, 1748.

COMPENDIUM FOR SELF-INSTRUCTION, WIDELY POPULAR IN ENGLAND AND AMERICA

THE
SCHOOLMASTER'S
ASSISTANT,
BEING A
COMPENDIUM OF ARITHMETIC,
BOTH
PRACTICAL AND THEORETICAL.
IN FIVE PARTS.
CONTAINING,

I. Arithmetic in whole Numbers, wherein all the common rules, having each of them a sufficient Number of Questions, with their answers, are methodically and briefly handled.

II. Vulgar Fractions, wherein several Things, not commonly met with, are distinctly treated of and laid down in the most plain and easy manner.

III. Decimals, in which, among other Things, are considered the Extraction of Roots; Interests, both simple and compound; Annuities, Rebate and Equation of Payments.

IV. A large collection of Questions with their Answers, serving to exercise the foregoing Rules, together with a few others, both pleasant and diverting.

V. Duodecimals, commonly called Cross Multiplication; wherein that sort of Arithmetic is thoroughly considered, and rendered very plain and easy, together with the Method of proving all the foregoing operations at once, by Division of several Denominations, without Reducing them into the lowest Terms mentioned.

The whole being delivered in the most familiar way of Question and Answer, is recommended by several eminent Mathematicians, Accomptants, and Schoolmasters, as necessary to be used in Schools by all teachers, who would have their Scholars thoroughly understand, and make a quick progress in Arithmetic.

TO WHICH IS PREFIXED,
AN ESSAY
ON THE EDUCATION OF YOUTH
HUMBLY OFFERED TO THE CONSIDERATION OF
PARENTS.

BY THOMAS DILWORTH,
Author of the New Guide to the English Tongue, Young Book-keeper's Assistant, &c.

NEW-YORK:
PRINTED BY T. & W. MERCEIN, 93 GOLD STREET
....
1815

LATE EDITION OF DILWORTH

The most popular arithmetic in the United States during and directly following the War of the Revolution.

ARAYA. Manila. A mathematical thesis defended at the University of the Society of Jesus in Manila. M.

1760 GEORGE FISHER, *The American Instructor.* New York. Printer: Gaine. 12th ed. Pp. v, (1), 378. N. Y. H. S.

1766 FISHER, *same.* New York. Printer: Gaine.

1770 FISHER, *same.* Philadelphia. Printer: Dunlap. Pp. v, 390, plate. 15th ed. A. A., N. Y. P.

Anon., *The Youth's Instructor . . . III Rules in Arithmetick.* Boston. Printer: Mein and Fleming. 152 pp. P.

1773 THOMAS DILWORTH, *The Schoolmaster's Assistant; being a Compendium of Arithmetic both practical and theoretical.* Philadelphia. Printer: J. Crukshank. Pp. (2), xiv, (10), 192; folded leaf; portrait. Hist. Soc. Penn.

1774 DANIEL FENNING, *The Ready Reckoner.* American edition of an English work. Germantown. Printer: Chr. Saur. A. A.

DANIEL FENNING, *Der Geschwinde Rechner, oder; des Händler's nutzlicher Gehülfe in Kauffung*, etc. Germantown. Printer: Christoph Saur. Largely tables, but strictly arithmetical. N. Y. P.

1775 *Thêses de Mathematiques qui seront soutenues au Seminaires de Quebec . . . . par* MM. . . . . PANET, PERRAULT, CHAUVEAUX. Quebec. N. Y. P.

1775 FISHER, *Instructor, or Young Man's Best Companion.* Burlington, N. J. Printer: Isaac Collins. Pp. xii, 372. Hist. Soc. Penn.

1778 FISHER, *same.* New York. Printer: Hugh Gaine.

1779 COCKER'S *Arithmetick.* Philadelphia. Reprint of an English work.

FISHER, *American Instructor.* Boston. Printer: Boyle and M'Dougall. Pp. vi, 378; folded plate. A. A., N. Y. P.

1781 DILWORTH, *The Schoolmaster's Assistant.* Philadelphia. A reprint of an English work. Apparently two editions: one printed and sold by R. Aitken, 220 pp., and the other by Jos. Crukshank, xiv, 5 l., 192 pp.; 1 folded leaf; 1 portrait. A. A.

1782 DEARBORN, *The Pupil's Guide.* Portsmouth. Printer: Daniel Fowle. 2 l.+16 pp. A. A.

1783 DEARBORN, reprint of above. Boston. Printer: B. Edes and Sons. Pp. vi, 26. M.

1784 DILWORTH, *The Schoolmaster's Assistant.* New York. Printer: Gaine. xiv, 5 l., 192 pp.; folded leaf; portrait. A. A.

DILWORTH, *same.* Philadelphia. Printer: Joseph Crukshank. Pp. (2), xiv, (10), 192; 1 folded leaf; 1 portrait. A. A.; Hist. Soc. Penn.

1784 An Arithmetical Card. Philadelphia. Sold by William Poyntell.

1785 THOMAS DILWORTH, *The Schoolmaster's Assistant.* Hartford. Printer: N. Patten. 200 pp. N. Y. P.

ALEXANDER MCDONALD, *The Youth's Assistant.* Norwich. Printer: John Trumbull. 102 pp. A. A.

FISHER, *The American Instructor.* New York. Printer: Hugh Gaine.

FISHER, *The Instructor; or, American Young Man's Best Companion.* Worcester. Printer: Isaiah Thomas. Pp. 384; frontispiece. A. A., N. Y. P.

1786 *Method of teaching rudiments of arithmetic.* Broadside (*i.e.*, single page). Providence. John Carter Brown Library.

1786 BONNYCASTLE, *The Scholar's Guide to Arithmetic.* Boston. Reprinted by John West Folsom. Pp. 200+2 leaves. M.

DILWORTH, *The Schoolmaster's Assistant.* Hartford. Printer: N. Patten. 9 l., pp. 9–199+1 bl.; portrait; folded leaf. A. A.

1786 LUDWIG HOECKER, *Rechenbuechlein.* Ephrata, Penn.

DANIEL FENNING, *The New Ready Reckoner.* Reprint of an English text. New York. Printer: Samuel Campbell.

FISHER, *Instructor*, etc. Worcester. Printer: Isaiah Thomas. 384 pp., 6 plates. N. Y. P.

FISHER, *American Instructor.* Philadelphia. Printer: J. Crukshank. Pp. xii, 372; 6 plates. A. A.

1787 DILWORTH, *Schoolmaster's Assistant.* Hartford. Printer: N. Patten. Pp. 384+frontispiece.
Advertised May 14, 1787.

THE

SCHOLAR's GUIDE

TO

ARITHMETIC:

OR

A complete Exercise-Book,

FOR THE

USE OF SCHOOL

WITH NOTES,

CONTAINING

The REASON of every RULE, demonstrated from the most simple and evident PRINCIPLES;

TOGETHER WITH

General THEOREMS for the more extensive USE of the SCIENCE.

By JOHN BONNYCASTLE,

Private TEACHER *of the* MATHEMATICS.

THE SECOND EDITION, corrected.

LONDON, Printed.

BOSTON: Re-printed by JOHN WEST FOLSOM, at the Corner of *Ann-street*. MDCCLXXXVI.

BONNYCASTLE'S ARITHMETIC

Bonnycastle's treatises, popular in America, covered practically the whole range of elementary mathematics.

FISHER. *American Instructor*. Philadelphia. Printer: J. Crukshank. Pp. xii, 372+frontispiece. A. A.

1788 NICHOLAS PIKE. *A New and Complete System of Arithmetic*, composed for the use of the citizens of the United States. Newburyport. Printed and sold by John Mycall. 512 pp. A. A., M., N. Y. P.

1788 THOMAS SARGEANT, *Elementary Principles of Arithmetic*, with their application to the trade and commerce of the United States of America. Philadelphia. Printer: Dobson and Lang. Pp. (4), 95, (1). A. A.

"A few copies intended for the use of young ladies are printed on fine paper and in a more elegant binding." A. A.

1788 SARGEANT, *A Synopsis of logarithmical arithmetic*. Philadelphia. Printer: Dobson and Lang.

SARGEANT, *Select Arithmetical Tables*. Philadelphia. Printer: Dobson and Lang. Single card.

GOUGH, *A Treatise of Arithmetic*. Ed. by Benj. Workman. Philadelphia. Printer: McCulloch. Pp. 370+1 leaf. A. A.

1789 BENJAMIN WORKMAN, *The American Accountant*. Philadelphia. Printer: John McCulloch. Based on the English work by John Gough. Pp. (2), (2), 224. A. A.

ALEXANDER MCDONALD, *The Youth's Assistant*. 2nd ed. Litchfield. Printer: Thomas Collier. 104 pp. P.

FENNING, *Ready Reckoner*. Reading. Printer: Benj. Johnson. 195 pp. A. A.

FENNING, *Der Geschwinde Rechner*. Reading. Printers: Johnson, Barton und Jungman. 195 pp. In collection of Mr. Wilberforce Eames.

JOHN GOUGH, *Treatise on Arithmetic*. Ed. by Benj. Workman. Boston.

1790 CONSIDER and JOHN STERRY, *The American Youth: being a new and complete course of introductory mathematics*. Providence. Printer: John Wheeler. Pp. 387, (1). A. A.; M.

DILWORTH, *Schoolmaster's Assistant*. Philadelphia. Printer: J. Crukshank. Pp. xiv, (10), 192; portrait. A. A.

BONNYCASTLE, *The Scholar's Guide.* New York.
DANIEL FENNING, *The Ready Reckoner.* Reading.

1791 ALEXANDER McDONALD, *The Youth's Assistant.* Providence.
DILWORTH, *Schoolmaster's Assistant.* Wilmington. Printer: Andrews, Craig, and Brynberg. Pp. xiv, (6), (2), (2), 192; folded table; portrait. A. A.

1792 FENNING, *Ready Reckoner.* Worcester. Printer: Thomas. Pp. (2), (2), 166. A. A.

1792 GORDON JOHNSON, *An Introduction to Arithmetic.* Springfield. Printer: E. W. Weld.
JOHN VINALL, *The Preceptor's Assistant, or Student's Guide:* being a systematical treatise of arithmetic. Boston. P. Edes, for Thomas and Andrews. 288 pp. A. A.
THOMAS DILWORTH, *Schoolmaster's Assistant.* New York. Printer: Gaine. Pp. xiv, (6), (2), (2), 192; folded table. A. A.

1793 THOMAS SARGEANT, *The Federal Arithmetician.* Philadelphia. Printer: Thomas Dobson. 264 pp.; 2 folded leaves. A. A.; M.
ERASTUS ROOT, *Introduction to Arithmetic.* Norwich, Conn.
NICHOLAS PIKE, *Abridgement of the new and complete system of arithmetic.* Newburyport. Printed by J. Mycall for Isaiah Thomas. Pp. i–iv, 5–371. A. A., M.
Another Copy . . . Printed for Isaiah Thomas, Worcester. Composed for the use and adapted to the commerce of the citizens of the United States . . . for the use of schools, and will be found to be an easy and sure guide to the scholar. A. A.
Pike's arithmetics were the first widely popular American works.

1793 DILWORTH, *Schoolmaster's Assistant.* "Latest Edition." New York. Printed by J. Allen. xvi, 3 l., 192 pp.; portrait. A. A.
DILWORTH, *Schoolmaster's Assistant.* "Latest Edition." New York. Printed by John Buel. xvi, 3 l., 192 pp.; portrait; folded leaf. A. A.
DILWORTH, *Schoolmaster's Assistant.* Philadelphia. Printer: Crukshank. xiv, 5 l., 192 pp.; portrait; folded leaf. A. A.

GORDON JOHNSON, *Introduction to Arithmetic*. 2nd ed. Springfield. Printer: J. R. Hutchins. 27 pp.+1 bl. A. A.

PHINEHAS MERRILL, *The Scholar's Guide to Arithmetic*. Exeter.

GOUGH, *American Accountant*. 2nd ed. by Benj. Workman, revised by R. Patterson. Philadelphia. Printer: Wm. Young. Pp. 220, (4). A. A.

Anon., *The Federal or New Ready Reckoner*. Chestnut Hill. Printer: S. Sower. 2 l.+56 l.+4 folded leaves+4 l. P.

1794 PHINEHAS MERRILL, *The Scholar's Guide to Arithmetic*. 2nd ed. Exeter. Printer: H. Ranlet. Pp. iv, 107. A. A.

FISHER, *The Instructor*. Walpole, N. H. Printer: Thomas and Carlisle. 384 pp., 5 plates. A. A.

1794 DANIEL FENNING. *The Ready Reckoner*. Newburyport. 11th ed. Printer: E. M. Blunt. 160 pp. N. Y. P.

JOHN TODD, ZACHARIAH JESS, WILLIAM WARING and JEREMIAH PAUL, *American Tutor's Assistant*. Philadelphia. 3rd ed. Printer: Z. Poulson, Jr. 2 l.+196 pp.+(imperfect copy). Hist. Soc. Penn.

1794 D. GREGORIUS BEDOYA. *Selectiones ex Elementis etiam tum Arithmeticae tum Algebrae utrius que etiam planae ac solidae Geometriae*. Sectionum Conicarum denum Trigonometriae, ac practicae Geometricae quas publico . . . . exhibet D. Gregorius Bedoya, Lima, Peru.

There were numerous other theses defended at the University in Lima, printed between 1768 and 1800.

Anon., *A short and easy guide to arithmetick*. Boston. Printer: Samuel Hall. John Carter Brown Library

1795 JOSEPH CHAPLIN, *The Trader's best companion*. Newburyport. Printer: Wm. Barrett. 36 pp. P.

PIKE. Abridgement, 2nd ed. Worcester. Printed by Isaiah Thomas. Apparently there was a second imprint of the same year. 348 pp. A. A., M.

Anon., *The Stranger's Assistant and Schoolboy's Instructor*. New York. A. A.

FENNING, *American Youth's Instructor*. Dover, N. H. Printer: Samuel Bragg. iv, 260 pp. A. A.

ERASTUS ROOT, *An Introduction to Arithmetick for the use of Common Schools.* Norwich. Printer: Thomas Hubbard. 105 pp.+1 bl. A. A., P.

1795 CONSIDER and JOHN STERRY, *A Complete Exercise Book in Arithmetic.* Norwich. Printer: John Sterry & Co. 120 pp. +1 l. A. A.

WM. WILKINSON, *The Federal Calculator and American Ready Reckoner.* Providence. Printer: Carter and Wilkinson. 64 pp. A. A.

1796 DONALD FRASER, *The Young Gentleman's and Lady's Assistant . . . including . . . Practical Arithmetic.* New York. Printer: T. &. J. Swords. 216 pp. A. A.

WORKMAN, *American Accountant.* Revised by Patterson. Philadelphia. 3rd ed. Printer: Wm. Young. 220 pp., 2 l. A. A.

GOUGH, *A Treatise of Arithmetic.* Ed. by Benj. Workman. Philadelphia. Printer: Wm. Young. 376 pp. A. A., M.

ERASTUS ROOT, *An Introduction to Arithmetick for the Use of common schools.* Norwich. 2nd ed. Printer: Thomas Hubbard. 106 pp. A.A.

ERASTUS ROOT, *An Introduction to Arithmetick for the Use of common schools.* Boston. Harvard College Library.

DILWORTH, *Schoolmaster's Assistant.* New York. Printer: Mott and Lyon. xii, 3 l., 192 pp.; frontispiece. A. A., N. Y. P.

DILWORTH, *Schoolmaster's Assistant.* Philadelphia. Printer: Crukshank. xiv, 4 l., 192 pp.; portrait; folded leaf. A. A.

DILWORTH, *Schoolmaster's Assistant.* Wilmington. Printed by Peter Brynberg. 3 l., 192 pp. A. A.

1796(?) DILWORTH, *Schoolmaster's Assistant.* Wilmington. Printer: Brynberg. 3 l., 192 pp. A. A.

1797 DILWORTH, *Schoolmaster's Assistant.* New London. Printer: Samuel Green. xvi, 3 l., 192 pp., 1 l.; 1 folded leaf; 1 portrait. A. A.

1797 TODD, JESS, WARING, and PAUL, *American Tutor's Assistant.* 3rd ed. Philadelphia. Printer: Z. Poulson. 200 pp. A. A.

1797 CHAUNCEY LEE, *American Accountant*. Lansingburg. Printer: W. W. Wands. 300+12 pp. list of subscribers + 1 plate. First to use the dollar sign in print. A. A., M.

FISHER, *The Instructor*, etc. Wilmington. Printer: Brynberg. xii, 360 pp. A. A.

JAMES NOYES, *The Federal Arithmetic*. Exeter.

DAVID KENDALL, *The Young Lady's Arithmetic*. Leominster, Mass. Printer: C. Prentiss. 44 pp. A. A.

WILLIAM MILNS, *American Accountant*. New York. Printer: J. S. Mott. Pp. (8)+320. A. A., M.

F. NICHOLS, *A Treatise of Practical Arithmetic and Bookkeeping*. Boston. Printer: Mann and Loring. 104 pp. A. A.

NICHOLAS PIKE, *A New and Complete System of Arithmetic*. 2nd ed. Worcester. Printer: Isaiah Thomas. 516 pp.+ an extra leaf numbered 5–6 with Preface by Isaiah Thomas. A. A.

FRANCIS WALKINGAME, *The Tutor's Assistant*. A new edition. Gainsborough. Printer: H. Mozley & Co. 180 pp.+ folded leaf. A. A.

1798 PETER THARP, *New and Complete System of Federal Arithmetic*. Newburgh. Printer: D. Denniston. 138 pp.+1 leaf. A. A.

NICHOLAS PIKE. *Abridgement*. Worcester. 3rd ed. Printer: Thomas. 352 pp. M.

SAMUEL TEMPLE, *A Concise Introduction to Practical Arithmetic*. Boston. Printer: Hall. 2nd ed., 118 pp. P.

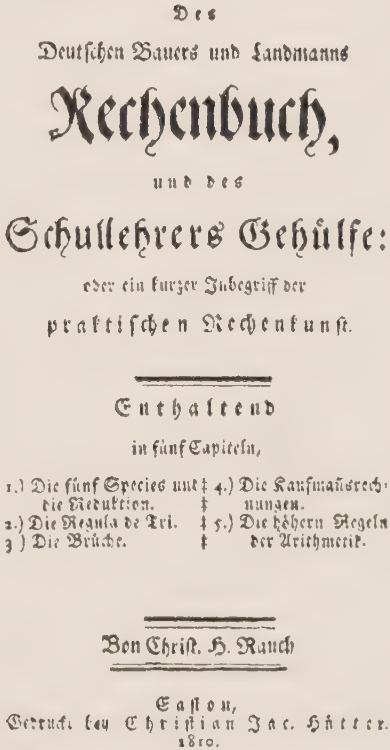
1799 DAVID COOK. *American Arithmetic*. New Haven. Printer: T. Green. Ed. of 1800 in Library of Congress may be first edition.

EZEKIEL LITTLE, *The Usher*, etc. Exeter. Printer: H. Ranlet. 240 pp. A. A.

DANIEL FENNING, *The Ready Reckoner*. Wilmington. Printer: Brynberg. 189+3 pp. 12th ed. Hist. Soc. Penn.

ZACHARIAH JESS, *The American Tutor's Assistant*. Baltimore. Printer: Bonsal and Niles. 2 l., 204 pp., 1 leaf errata. L. C.

1799 (?) DILWORTH, *The Schoolmaster's Assistant.* Printed and sold by Bonsal and Niles. Also sold at their book store, Baltimore. 3 l., 192 pp.; folded leaf; portrait. A. A.


Des

Deutschen Bauers und Landmanns

Rechenbuch,

und des

Schullehrers Gehülfe:

oder ein kurzer Inbegriff der

praktischen Rechenkunst.

Enthaltend

in fünf Capiteln,

1.) Die fünf Species und die Reduktion.
2.) Die Regula de Tri.
3.) Die Brüche.
4.) Die Kaufmañsrechnungen.
5.) Die höhern Regeln der Arithmetik.

Von Christ. H. Rauch

Easton,
Gedruckt bey Christian Jac. Hütter.
1810.

GERMAN ARITHMETIC

Strange title for an American arithmetic: *The German Peasant's and Farmer's Arithmetic and Schoolmaster's Assistant.*

Arithmetics in German were published in St. Louis from 1850 to 1860, and doubtless more recently in other parts of the United States.

EARLY CANADIAN ARITHMETICS

1809 JEAN ANTOINE BOUTHILLIER, *Traité d'Arithmétique pour l'usage des Écoles.* Quebec. Printer: John Neilson. 3 l., 144 pp. 2nd ed. 1829. P., L. C.

REV. JOHN STRACHAN, *A concise introduction to practical arithmetic for the use of Schools.* Montreal, 1809. Toronto Public Library.

1816 MICHEL BIBAUD, *L'arithmétique en quatre parties . . . vulgaire, . . . marchande . . . scientifique . . . curieuse . . .* Montreal. Printer: N. Mower. iv, 199 pp. 2nd ed. widely used, 1832.

1836 CASIMIR LADREYT, *Nouvelle arithmétique.* Montreal. Printed by author. viii, 120 pp.

1836 JOS. LAURIN, *Traité d'Arithmétique et d'Algèbre.* Quebec. 206 pp.

KENTUCKY AND WEST OF THE MISSISSIPPI RIVER

1804 JESSE GUTHRIE, *The American Schoolmaster's Assistant.* Lexington, Kentucky. 231 pp.
A copy of the 1810 edition is in the Library of Congress.

1822 RÉNÉ PAUL, *Elements of Arithmetic.* St. Louis, Mo. Printer: Ford and Orr. 18 mo., 160 pp.

1826 J. STOCKTON, *The Western Calculator.* Fourth edition, 1826. Pittsburgh. Printer: Eichbaum and Johnston. 204 pp. L. C.

1828 Anon., *He helu kamalii, o ke aritemetika.* Ohau, Hawaii, 1828. (Missionary Press.) Second edition, 1832.

1836 *Tabla para los niños que empiezan contar.* Broadside. Monterey, California.

1836 *Tablas para los niños*, etc. Monterey. Printer: A. Zamorano.

The largest collections of early American arithmetics are to be found in the Library of Congress (L. C.), New York Public Library (N. Y. P.), American Antiquarian Society (A. A.), and in the private library of Mr. George A. Plimpton, New York City. The University of Michigan (M.) has a fair collection.

The writer is particularly indebted to Dr. Clarence S. Brigham of the American Antiquarian Society for suggestions and notes on books; to Dr. Wilberforce Eames of the New York Public Library for information touching many points of this bibliography; to Dr. Lawrence C.

Wroth of the John Carter Brown Library for notes; and also to the librarians at the other places mentioned.

The Hawaiian arithmetic appears to be the first published in the New World west of St. Louis. The information concerning the first edition was very kindly given by the Reverend Howard M. Ballou of Honolulu.

Bibliography for Supplementary Reading

D. E. Smith, *Rara Arithmetica.* *See* Bibliography, Chapter II.

Augustus De Morgan, *Arithmetical Books.* *See* Chapter II.

J. O. Halliwell, *Rara Mathematica.* *See* Chapter II.

Florian Cajori, "The Teaching and History of Mathematics in the United States." Washington, Bureau of Education, *Circular* No. 3, 1890.

Walter S. Monroe, "Development of Arithmetic as a School Subject." Washington, Bureau of Education, *Bulletin* No. 10, 1917.

David Eugene Smith, *The Sumario Compendioso of Brother Juan Diez.* Boston, Ginn and Co., 1921.

Robert Steele, "The Earliest Arithmetic in English." London, *Early English Text Society*, Extra Series No. cxviii, 1922.

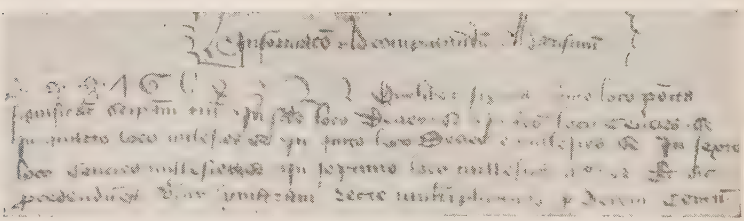
George E. Littlefield, *Early Schools and Schoolbooks of New England.* Boston, The Club of Odd Volumes, 1904.

Susan R. Benedict, "A Comparative Study of the Early Treatises introducing into Europe the Hindu Art of Reckoning." Thesis, University of Michigan. Published privately, 1916.

CHAPTER IV

THE FUNDAMENTAL OPERATIONS IN EARLY ARITHMETIC EMPLOYING NUMERALS

Fundamental operations. Today we speak of the four fundamental operations of arithmetic without hesitation as to the number of the operations. The early works on our system of arithmetic include frequently seven, eight, or nine subdivisions. Numeration or notation continued to be called a fundamental operation until the nineteenth century; doubling and halving as separate operations appear in nearly all the treatises before the fifteenth century, and in many up to the seventeenth century; extraction of roots, square and cube root, was regarded as fundamental in the Arabic treatises up to the



British Museum, Add. MS. 24059 fol. 22b.

NUMERATION IN A FIFTEENTH-CENTURY MANUSCRIPT

The first line of the text begins with the numerals 0, 9, 8, . . . 1, in reverse order, possibly due to the fact that Arabic writing proceeds from right to left. The text continues: "Quelibet figura primo loco posita significat seipsam, tamen in secundo loco Decies seipsam . . . . " This means that any digit in the first or units' place signifies so many units, in the second place so many tens, and so on.

The title is "Informacio ad computandum Algorismi," or "Information concerning the Computation by Algorism."

twelfth century, and in the popular works of Alexandre de Ville Dieu and Sacrobosco; progressions also are

included by these two writers, by Recorde, and by many others. The Hindu arithmeticians do not include doubling and halving, but several list among the fundamental operations many topics on application to practical affairs. Mahavir in the ninth century lists only eight operations: squaring, square root, cubing, cube root, summation (addition), multiplication, division, and subtraction, the final two applied to series.

> "Septem sunt partes, non plures, istius artis:
> Addere, subtrahere, duplare, dimidiare,
> Sextaque diuidere, sed quinta multiplicare;
> Radicem extrahere pars septima dicitur esse.

"Here telles that ther ben 7 spices or partes of this craft. The first is called addicion, the secunde is called subtraccion. The thryd is called duplacion. The 4 is called dimydicion. The 5 is called multiplicacion. The 6 is called diuision. The 7 is called extraccion of the Rote. What all these spices bene hit schalle be tolde singillatim in here caputule."

This is Alexandre de Ville Dieu's notion of the "species" or operations, and also the notion of his English commentator. Robert Recorde, three centuries later, says

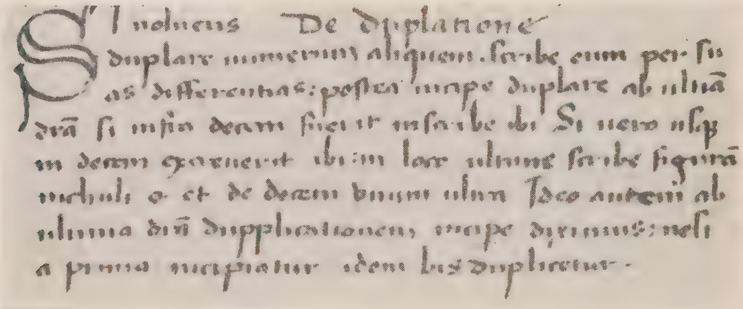
"*There are reckoned commonly seven parts or works of it.*

"Numeration, Addition, Subtraction, Multiplication, Division, Progression, and Extraction *of roots: to these men adde* Duplication, Triplation, and Mediation."

Nicholas Pike states that there are "five principal or fundamental Rules, *viz.* Notation or Numeration, Addition, Subtraction, Multiplication and Division." With the occasional inclusion of the extraction of roots and later with the exclusion of notation, this list is fairly typical of the fundamental operations as listed in works in English from the eighteenth century to the present day.

Addition and Subtraction

Addition. The operations of addition and substraction are so elementary that much variation in procedure from



Paris, MS. Latin 10252, Bibliothèque Nationale

DUPLICATION

As given in a fifteenth-century copy of a twelfth-century treatise on the Hindu art of reckoning.

early times to the present would not be expected. Nevertheless addition continued for centuries with marked peculiarities, testifying to the strength of the influence of the abacus reckoning upon later methods in arithmetic. In subtraction three lines of procedure are widely followed even today, while yet further variations are found in early treatises.

Upon the abacus or upon lines not more than two numbers can conveniently be written; in summation on an abacus the lower number is naturally combined with the upper number in such a way that finally the sum alone remains upon the board. Precisely this method was taught with the written numerals, even though these are well adapted to the summation of a series of numbers, with the numbers to be summed retained upon the paper.

Sacrobosco follows the early tradition in this respect. His version is given by the English translator of the fifteenth century, as follows:

"In addicioun, 2 ordres of figures and 2 nombres ben necessary, that is to sey, a nombre to be addede and the nombre whereto the addicioun sholde be made to. . . .

"Therefor, yf thow wilt adde nombre to nombre, write the nombre whereto the addicioun shalle be made in the omest [*i.e.*, highest] ordre by his differences, so that the first of the lower ordre be vndre the first of the omyst ordre, and so of others. That done, adde the first of the lower ordre to the first of the omyst ordre. And of suche addicioun, other there growith thereof a digit, An article, other a composede. If it be digitus, In the place of the omyst shalt thow write the digit excrescying, as thus:

The resultant	2
To whom it shal be addede	1
The nombre to be addede	1" [1]

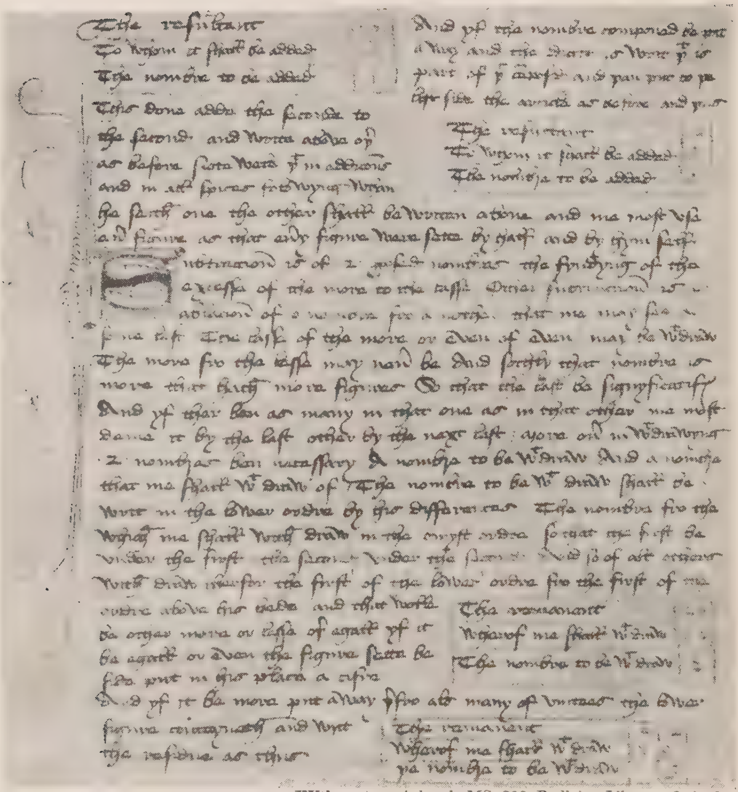
The Latin and other early versions almost invariably indicate to "add a number to a number," and many of them indicate also that in the process both the numbers vanish, leaving the sum in the place of the larger number. To begin at the left to add was almost as common as to begin at the right. The successive stages of the addition of 826 and 483 are represented graphically, as follows:

826	829	909	1309
483	48	4	

Leonard of Pisa (1202) departs from the custom of only two addends, and also introduces the novelty of writing the sum above the addends.

With the appearance of printed arithmetics in the fifteenth century addition assumed the modern form.

[1] Steele, *loc. cit.*, p. 35.



XVth century Ashmole MS. 396, Bodleian Library, Oxford

ADDITION AND SUBTRACTION IN "THE ART OF NOMBRYNG"

Above at the right is an addition example in which the sum, "The resultant," appears at the top; "To whom it shall be addede" is 8, and "The nombre to be addede" is 4.

With the decorative S begins the treatment of subtraction, as follows:

"Subtraccioun is of 2 propocede nombres the fynding of the excesse of the more to the lesse. Other subtraccioun is ablacioun of o nombre fro another that me may see a some left."

The remainder, "remanent," is written above in the two problems at the foot of the page.

Subtraction. Subtraction in the early algorisms reveals also peculiarities which follow the procedure on an abacus. The remainder replaces the minuend, both minuend and subtrahend disappearing. Even Pike in 1788 retains two peculiarities found in several of the algorisms: "If the lower figure be greater than the upper, borrow ten, and subtract the lower figure therefrom: to this difference add the upper figure, which, being set down, you must add one to the tens' place of the lower line for that which you borrowed." This method of "borrowing above" and "paying back below," common in England from Shakespeare's day (Recorde, Baker) almost to the present time, has its advocates in America today.

Austrian method. The strictly addition procedure in subtraction is mentioned in the *Handbuch der Mathematik* by Bittner, published at Prague in 1821; the method is explained in Solomon's *Lehrbuch der Arithmetik und Algebra*, Vienna, 1849. In America this has been known as the Austrian method, and its use is recommended to primary teachers in the courses of study of several large school systems.

The procedure is as follows:

$$\begin{array}{r} 826 \\ \underline{483} \\ 343 \end{array}$$

Think of the number which added to 483 will give 826; **3** added to 3 gives 6; **4** added to 8 gives 12; write down **4** and mentally carry **1** to the next 4, making 5; **3** added to 5 makes 8. **343** is the number which added to 483 gives 826.

The "check by nines." The check upon subtraction by addition, and vice versa, is particularly recommended by the earliest writers on arithmetic. The further "check

by nines" upon addition, subtraction, and other operations was quite as common among earlier writers. The "check by nines" depends upon the fact that 10, 100, 1000, . . . each when divided by 9 has 1 as remainder; in consequence a number like 6724 when divided by 9 will have the remainder $6+7+2+4$ or "casting out" 9 twice the remainder will be 1. In other words, the remainder when dividing a number by 9 is obtained by taking the sum of the digits and, if necessary, casting out any multiple of 9. If a number whose remainder when divided by 9 is 1 is added to one whose remainder is 6, it is obvious that the resulting number when divided by 9 will have the remainder $1+6$ or 7. Thus

$$\begin{array}{l} 6724 \text{ rem. } 1 \\ \underline{1221} \text{ rem. } 6 \\ 7945 \text{ rem. } 7 \end{array}$$

This is indicated in many old texts by a diagram at the side.

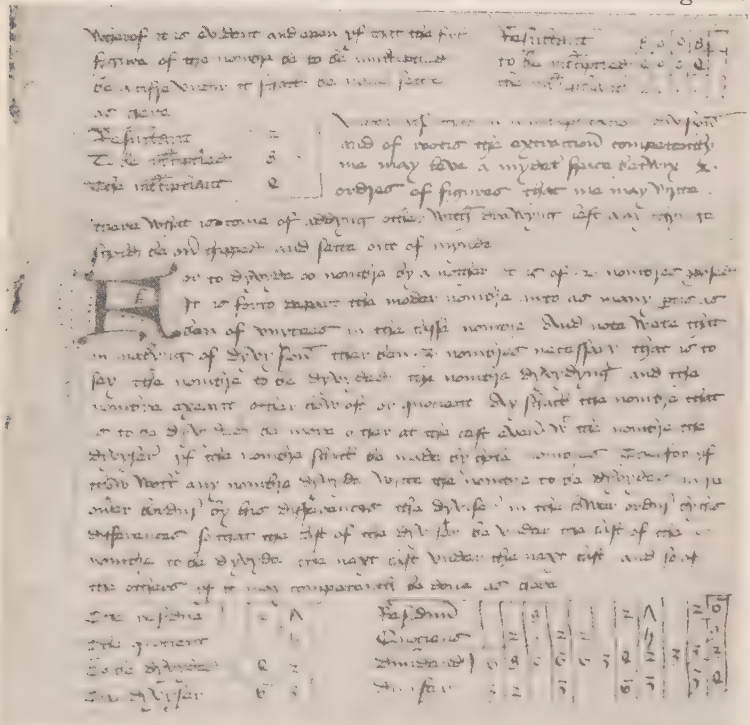
Obviously, there are somewhat analogous rules, with the proper changes, to apply the "check by nines" to the other fundamental operations.

Multiplication

Early methods of multiplication. The operation of multiplication invites a variety of methods of treatment. Several of the early methods employed are instructive and worthy of somewhat detailed treatment.

The method in most common use in Europe in the early algorisms corresponds precisely to the procedure indicated, but not fully explained, in the early Hindu treatises.

Brahmagupta states that "the multiplicand is repeated like a string for cattle as often as there are integrant



MULTIPLICATION AND DIVISION IN "THE ART OF NOMBRYNG"

The product or "resultant" is placed above in the two illustrative problems in multiplication.

In the final example on this page, under division, four separate problems are given: 680 ÷ 32; 66 ÷ 3; 342 ÷ 63; and 332 ÷ 34, which last has the quotient 9, and the "residuum," or remainder, 26.

portions[1] in the multiplier, and is severally multiplied by them." Mahavir says that multiplier and multiplicand are placed "in the manner of the hinges of a

[1] Meaning digits.

door," and Sridharacarya adds to this "multiply in order, directly or inversely, repeating the multiplier each time." The method indicated is explained in *The Crafte of Nombrynge*, one of the two earliest discussions in English.

"Here begynnes the Chaptre of multiplication, in the quych thou must know four thynges. First, qwat is multiplicacion. The secunde, how mony cases may hap in multiplicacion. The thryde, how mony rewes of figures there most be. The 4 what is the profet of this craft."[1]

Before giving the complete explanation, the writer interjects the multiplication table in triangular form from 1×1 up to 9×9, as being necessary in the multiplication.

"Lo an Ensampul here folowynge."[2]

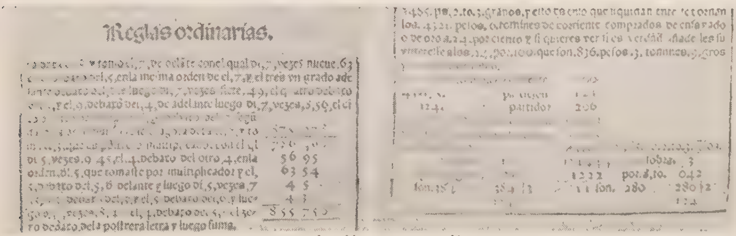
2465 *a.* 232	464465 *b.* 232	1 82 464865 *c.* 232
11 121 828 *d.* 464825 232	11 110 1211 8285 *e.* 464820	

2465 is to be multiplied by 232 (multiplier). In figure *b* 232 has been multiplied by 2 (for 2000), and the product 464 (for 464000) written in the same line with 2465, of which the 2 has disappeared in the final step. Next 232 is moved over one space and multiplication of 232 by 4 (400) follows. The partial products (8 for 80000, 12 for 12000) are written above, and finally 8 for 800 (400×2) takes the place of the multiplying digit 4. Then

[1] Steele, *loc. cit.*, p. 21. [2] Steele, *loc. cit.*, pp. 24–25.

232 is moved again one place, until the unit 2 falls under the 6 of the multiplicand. The partial products again are written above. Finally 232 is moved again so that the units' place 2 comes under the sole remaining digit of the multiplicand, 5, and the partial products are again written above.

After the partial products are written they are summed,


Reglas ordinarias.

JUAN DIEZ FREYLE'S "SUMARIO," MEXICO, 1556

Multiplication of 875 by 978 with all partial products completely written in column form.

At the right, division by the "scratch method" of 432175 by 124. This method is explained below, under Division.

here from left to right, and 571880 appears as the final result.

Variations. Some writers of the twelfth century combine as they go along; in some the figures do not disappear but are "scratched out," giving rise to the name "scratch method."

A variation of the above process is given by Juan Diez Freyle in the first multiplication example to appear in the New World. Freyle herein probably follows Spanish arithmeticians of the early sixteenth century; the variation is found in Paciuolo's work and in numerous others.

Another method involving all separate partial products is introduced by Paciuolo as the "lattice work" or

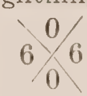
"jealousy" method. The *jalouse* or "lattice work" is that screen behind which ladies are accustomed to stand to observe without being observed. The derivation from the same word as our "jealous" is obvious. In this device as applied to multiplication the units, tens, and hundreds appear in the same *diagonal* row and are combined diagonally. Napier, the inventor of logarithms, introduced a variation of this method by having the multiplication table written on "rods" of wood or bone, whence called "Napier's bones." Recently an English writer again suggests the use of Napier's rods in arithmetic.

Beginnings of modern methods. Our present method of multiplication appears in the Treviso arithmetic of 1470, in Calandri in 1491, in Paciuolo, and in printed arithmetics from that day to this.

Early German and Italian arithmetics sometimes presented a "lightning" method of multiplication in which only the digits of the product are written in one line below multiplier and multiplicand. The illustration given by Paciuolo in 1494 indicates that the units' digit from the product of the units is written down, the tens' digit being kept "in the heart" or "in the hand" as Leonard of Pisa expressed it in 1202; then the two products giving tens are combined with this possible tens' digit from the product of the units; the three products involving hundreds (*e.g.* units by hundreds, twice, and tens by tens) are combined, and so on.

The following illustrates the "lightning" method:

$$\begin{array}{r} 456 \\ 456 \\ \hline 207936 \end{array}$$



The only units' digit is obtained from 6×6, giving 6 to be put down and 3 to be carried; the tens' digit is obtained from 6×5, twice, to which 3 is to be added; the hundreds' digit is obtained from 6×4, 6×4, and 5×5 with 6 carried over, etc. A German writer adds: "It takes much head."

At the right is indicated the check by nines, 6 and 6 being the remainders when the factors are divided by 9; 0 that of the product.

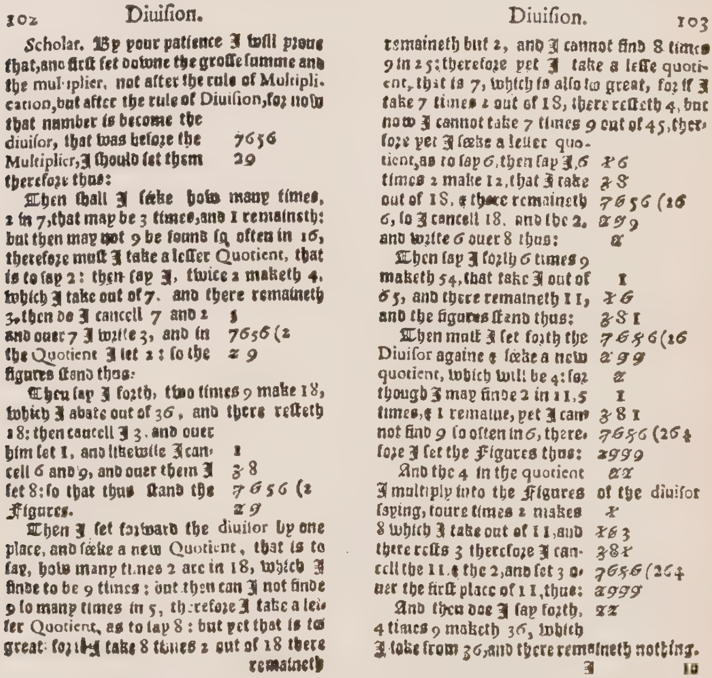
A notable desirable modern innovation consists in beginning multiplication with the highest and thus most significant figures of the multiplier. This is particularly useful in computations with decimals when only a limited number of places in the product are significant. As early as 1592 this method was used by Jobst Bürgi in an unpublished manuscript on arithmetic. The only necessary variation from our method is the reversal of the order of multiplication; however, with decimal fractions this method permits one more easily to neglect the non-significant digits.

$$
\begin{array}{lll}
232 & \qquad & 2.32 \\
\underline{624} & & \underline{6.24} \\
1392 & & 1392 \\
464 & & 46 \\
\underline{928} & & \underline{9} \\
144768 & & 14.47
\end{array}
$$

If 2.32 and 6.24 represent measurements, 14.47 is as accurate as the measurements justify; equally, if 232 and 624 represent the dimensions of a rectangle, the final 68 in 144,768 has no real significance in the area.

Division

"Scratch" method of division. The Hindus undoubtedly worked arithmetical problems on a board strewn with sand. With this arrangement it was more convenient



102 Diuision.

Scholar. By your patience I will proue that, and first set downe the grosse summe and the multiplier, not after the rule of Multiplication, but after the rule of Diuision, for now that number is become the diuisor, that was before the Multiplier, I should set them therefore thus:

7656
29

Then shall I seeke how many times, 2 in 7, that may be 3 times, and 1 remaineth: but then may not 9 be found so often in 16, therefore must I take a lesser Quotient, that is to say 2: then say I, twice 2 maketh 4, which I take out of 7, and there remaineth 3, then do I cancell 7 and 2 and ouer 7 I write 3, and in the Quotient I set 2: so the figures stand thus:

Then say I forth, two times 9 make 18, which I abate out of 36, and there resteth 18: then cancell I 3, and ouer him set 1, and likewise I cancell 6 and 9, and ouer them I set 8: so that thus stand the Figures.

Then I set forward the diuisor by one place, and seeke a new Quotient, that is to say, how many times 2 are in 18, which I finde to be 9 times; but then can I not finde 9 so many times in 5, therefore I take a lesser Quotient, as to say 8: but yet that is too great: for if I take 8 times 2 out of 18 there remaineth

Diuision. 103

remaineth but 2, and I cannot find 8 times 9 in 25: therefore yet I take a lesse quotient, that is 7, which is also too great, for if I take 7 times 2 out of 18, there resteth 4, but now I cannot take 7 times 9 out of 45, therefore yet I seeke a lesser quotient, as to say 6, then say I, 6 times 2 make 12, that I take out of 18, & there remaineth 6, so I cancell 18, and the 2, and write 6 ouer 8 thus:

Then say I forth 6 times 9 maketh 54, that take I out of 65, and there remaineth 11, and the figures stand thus:

Then must I set forth the Diuisor againe & seeke a new quotient, which will be 4: for though I may finde 2 in 11, 5 times, & 1 remaine, yet I cannot finde 9 so often in 6, therefore I set the Figures thus:

And the 4 in the quotient I multiply into the Figures of the diuisor saying, foure times 2 makes 8 which I take out of 11, and there rests 3 therefore I cancell the 11, & the 2, and set 3 ouer the first place of 11, thus:

And then doe I say forth, 4 times 9 maketh 36, which I take from 36, and there remaineth nothing.

I 10

ROBERT RECORDE'S "THE GROUNDE OF ARTES," *c.* 1542
An explanation of the scratch method, 7656÷29.

to erase the intermediate digits involved in any calculation, replacing them gradually by the figures of the final result. Some of the methods of multiplication explained in the preceding section doubtless had their origin in the sand table. In division a method bespeaking

such an origin was almost exclusively used in Europe until well toward the end of the fifteenth century, and

116 Diuision.

Diuide 7890 by 33.

First set them thus, then bring the Diuisor vnder 78, and see how oft it is there found, which is twice, and therefore set 2 in the quotient, by which multiply the Diuisor 33, and set the product 66 vnder 78, and subduct it out of it thus.

33) 7890 (2

Then bring the next figure 9 downe, and set it with the Remainer 12, it maketh 129, and remouing the Diuisor 33 thereto, enquire how often 33 is contained in 129, and I finde it but thrice, (though at the first it made a shew of more) therefore set 3 in the Quotient, and multiplying 33 by 3, set the product vnder 129, subducting that product out of the number aboue, and proceed as before.

$$
\begin{array}{r}
33)\,7890\,(239\tfrac{1}{33} \\
\underline{66}\phantom{90\,(239\tfrac{1}{33}} \\
129\phantom{0\,(239\tfrac{1}{33}} \\
\underline{99}\phantom{0\,(239\tfrac{1}{33}} \\
300\phantom{\,(239\tfrac{1}{33}} \\
\underline{297}\phantom{\,(239\tfrac{1}{33}} \\
3\phantom{\,(239\tfrac{1}{33}}
\end{array}
$$

Then shall you finde the diuisor 9 times in the Remainer, therefore setting 9 in the Quotient, multiply, and subduct as before, and at the last you shall finde onely 3 remaining, which must be set aboue a line after the Quotient, and the Diuisor vnder, as aboue appeareth.

Scholar. Is there no more difficulty in the

"THE GROUNDE OF ARTES"

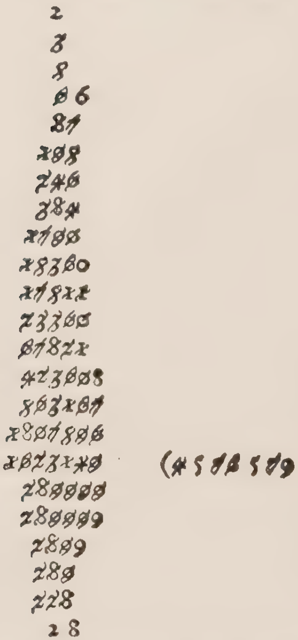
Our present method of division as taught by Recorde. The remainders and partial products were frequently printed in vertical columns as they appear here.

It is customary for authors to attach the blame for this type of error to the printer.

continued there in popular use until the nineteenth century. The figures which were deleted upon the sand table were scratched across in writing and thus came, as we have said above, the name "scratch" method, being usually applied to division. The printer found it necessary to have always two fonts of numerals, and his

objections to the method assisted in banishing it from printed books.

Other early methods. In the diagrams taken from Recorde's *Grounde of Artes*, 7656 is divided by 29; the third diagram shows the remainder 1856 after 58(00) or



CLAVIUS (KLAU) EPITOME ARITHMETICAE PRACTICAE, ROME, 1583

Father Christopher Clavius made long division longer by illustrating the process of correction after making an error in the assumed quotient.

Here 1623149 is divided by 2899 with a quotient of 559 and remainder 2608.

The printers objected to the method as it required two different sets of type for the numerals. Their influence assisted in banishing the awkward method.

2(00)×29 has been subtracted. Then the divisor 29 is moved one place to the right, the 9 on the upper line to bring the operation within reasonable bounds.

In the older treatises the divisor moves one space at a time to the right, the dividend disappears by subtracting the partial products directly from it, and the quotient digits are written in horizontal line, each one in the

same vertical column with the units' digit of the divisor as it moves.

7656	3656	1856	1856	
29	9		29	
	2	2	2	
1856	656	116	116	116
29	9		29	29
26	26	26	26	264

"And note wele that me may not withedraw more than .9. tymes nether lasse than ones. Therfor se how oft the figures of the lower ordre may be withdraw fro the figures of the ouerer, and the nombre that shewith the quocient most be writ ouer the hede of that figure, vnder the whiche the first figure is, of the dyviser; And by that figure me most withedraw alle other figures of the lower ordir and that of the figures aboue thaire hedis. This so done, me most sette forwarde the figures of the diuiser by o (one) difference towardes the right honde and worche as before; and thus:—

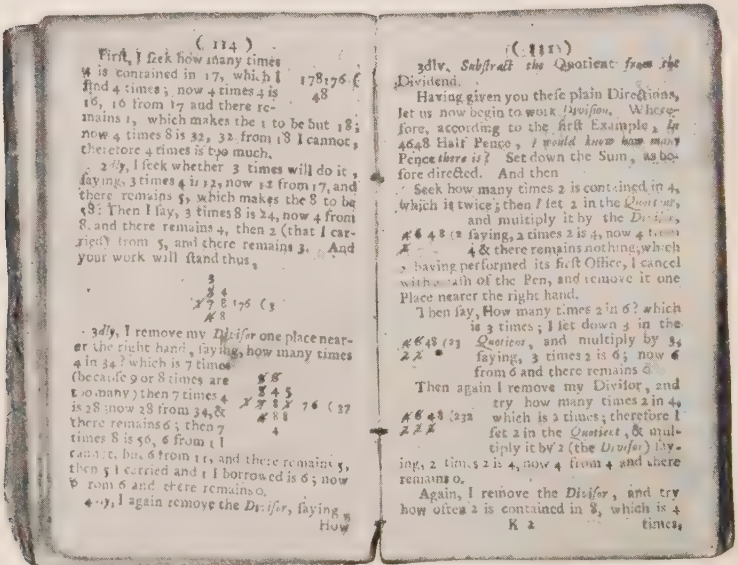
The residue						1	2
The quocient				2	0	0	4
To be dyvydede	8	8	6	3	7	0	4
The dyvyser	4	4	2	3"[1]			

A consideration of the above schematic form will reveal that this is the exact reverse of the early method of multiplication (*see* page 108); further that the remainders 18, 11, and 0 are precisely those which the present method of division exhibit.

Notwithstanding the appearance of our method of division in the first printed arithmetic, in Calandri, and in Paciuolo, and in the popular arithmetics of the sixteenth century, the scratch method was widely taught

[1]Steele, *loc. cit.*, pp. 45–46.

well into modern times, being included in the first separate arithmetic published in the United States, that by Hodder. The scratch method appeared in the earlier editions of Bradford's *The Young Man's Companion*, but was eliminated in the 1728 edition. Christopher Clavius,


(114)

First, I ſeek how many times 4 is contained in 17, which I find 4 times; now 4 times 4 is 16, 16 from 17 and there remains 1, which makes the 8 to be but 18; now 4 times 8 is 32, 32 from 18 I cannot, therefore 4 times is too much.

2dly, I ſeek whether 3 times will do it, ſaying, 3 times 4 is 12, now 12 from 17, and there remains 5, which makes the 8 to be 58: Then I ſay, 3 times 8 is 24, now 4 from 8, and there remains 4, then 2 (that I carried) from 5, and there remains 3. And your work will ſtand thus,

3dly, I remove my *Diviſor* one place nearer the right hand, ſaying, how many times 4 in 34? which is 7 times (becauſe 9 or 8 times are too many) then 7 times 4 is 28; now 28 from 34, & there remains 6; then 7 times 8 is 56, 6 from 1 I cannot, but 6 from 11, and there remains 5, then 5 I carried and 1 I borrowed is 6; now 6 from 6 and there remains 0.

4thly, I again remove the *Diviſor*, ſaying How

(115)

3dly. *Subſtract the* Quotient *from the* Dividend.

Having given you theſe plain Directions, let us now begin to work *Diviſion*. Wherefore, according to the firſt Example, *In* 4648 Half Pence, *I would know how many* Pence *there is?* Set down the Sum, as before directed. And then

Seek how many times 2 is contained in 4, which is twice; then I ſet 2 in the *Quotient*, and multiply it by the *Diviſor*, ſaying, 2 times 2 is 4, now 4 from 4 & there remains nothing; which having performed its firſt Office, I cancel with a daſh of the Pen, and remove it one Place nearer the right hand.

Then ſay, How many times 2 in 6? which is 3 times; I ſet down 3 in the *Quotient*, and multiply by 3, ſaying, 3 times 2 is 6; now 6 from 6 and there remains 0.

Then again I remove my Diviſor, and try how many times 2 in 4, which is 2 times; therefore I ſet 2 in the *Quotient*, & multiply it by 2 (the *Diviſor*) ſaying, 2 times 2 is 4, now 4 from 4 and there remains 0.

Again, I remove the *Diviſor*, and try how often 2 is contained in 8, which is 4 times,

K 2

BRADFORD'S "THE YOUNG MAN'S COMPANION," NEW YORK, 1710

Here is shown the scratch method in both short and long division, later replaced in other editions by our present method.

the popular textbook writer who finally effected the reform of the calendar, explains (1574) with praise the method which we use, but constantly employs the scratch method.

Austrian method. Division without writing of the remainders appears in the work of Clavius and in the American arithmetic of Isaac Greenwood (1729). Combined with the additive method of subtraction, this

ARITHMETICK.

reaſon of the *Periods* ſo plac'd in this *Example.* And having thus particularly conſider'd the *Manner*, I proceed to the Reaſon of theſe Operations; which will be very obvious to any One that ſhall conſider a little, the following *Form of Expreſſing* the ſame *Example.*

Diviſor	*Dividend*	*Quotient*
8)	6 8 5 5 2	(8000 The Firſt *Quotient.*
Subſtract	4 0 0 0	The *Product* of the *Diviſor* into the *Quotient*, viz. 8 into 8000; for the *Quotient* Figure is always of the Value of the Figure, under which the *Units* Place of its *Product* ſtands.
Diviſor 8)	4 5 5 2	(500 Second *Quotient* Figure;
Subſtract	4 0 0 0	(Being the *Product* of 8 into 500.
Diviſor 8)	5 5 2	(60 Fourth *Quotient* Figure;
Subſtract	4 8 0	(The *Product* of 8 into 60.
Diviſor 8)	7 2	(9 Laſt *Quotient* Figure;
Subſtract	7 2	(The *Product* of 8 into 9.
Remains	0 0	Now the *Sum* of all theſe ſeveral

Quotients Viz. 8000 + 500 + 60 + 9 = 8569.

EXAMPLE II.

To divide 590624922 by 7563. This is perform'd as follows.

```
Diviſor  Dividend   Quotient
 7563)   590624922  (78094
         52941....
         -----
          61214
          60504
          -----
            71092
            68067
            -----
             30252
             30252
             -----
             00000
```

The

LONG DIVISION AS TAUGHT BY ISAAC GREENWOOD IN 1729

constitutes the Austrian method of division. Theoretically, and undoubtedly practically if taught in the ele-

38 *Diviſion.* Chap. V.

and try how many times 2 in 4, which is two times, therefore I ſet 2 in the *Quotient*, and multiply it by 2 (the Diviſor) ſaying 2 times 2 is 4, now 4 from 4, and there remains 0.

4648 (232
222

Again, I Remove the Diviſor, and try again how often 2 is contained in 8, which is 4 times, I ſet 4 in the *Quotient* & multiply it by 2, ſaying, 4 times 2 is 8 : now 8 from 8, and there remains 0.

4648 (2324
2222

Another Example with one Figure.

Suppoſe there is 398 Pounds to be equally Divided between 6 men, the Demand is what each man muſt have?

Firſt, I ſet down the Dividend 398, & 6 (the Diviſor) under the 9 thus, becauſe I cannot take 6 out of 3.

398 (6
6

Then I try how many times 6 I can have in 39, which is 6 times, I place 6 in

Chap. V. *Diviſion.* 39

in the Quotient beyond the crooked line, ſaying, 6 times 6 is 36; now 36 from 39, and there remains 3, which I ſet down over the 9, and cancel the 39 & 6 my Diviſor, thus,

3
398 (6
6

Again, I remove my Diviſor to the next Place under 8, and ſeek how many times 6 I can have in 38, which is alſo 6 times, I ſet 6 in the Quotient, ſaying, 6 times 6 is 36, 36 from 38, and there remains 2, which 2 I ſet over the 8, and cancel the 6 thus;

32 (2
398 (66
66

So that every man muſt have 66 *l.* and 2 *l.* over, which I may turn into Pence, and divide alſo by 6, and the Quotient will be 80 Pence, which is in all 66 Pound 6 Shillings and 8 Pence a-piece.

This order I obſerve to Divide by one Figure; but if the Diviſor do conſiſt of more Figures than one, I muſt take the firſt Figure of the Diviſor no oftner out of the Dividend D 2 *than*

HODDER'S "ARITHMETICK," BOSTON, 1719

A labored explanation of division of 4648 by 2 by the scratch method.

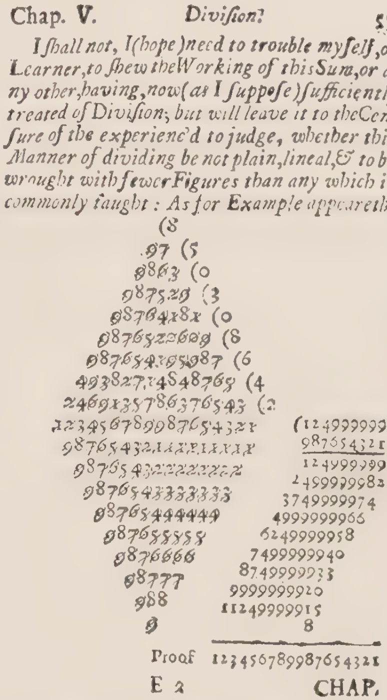
mentary school, the method has much to commend it.

$$\begin{array}{r|l} & 264 \\ \hline 29 & 7656 \\ & 18 \\ & 11 \\ & 0 \end{array}$$

2×9, 18, $8+18$, 26; put down **8** and carry mentally 2; 2×2, 4; $4+2=6$; $6+1=7$. The remainder 18 appears

below. 6×9, 54; $+1$, 55; 6×2 is 12; $+5$ is 17; $+1$ is 18; 4×9 is 36; $+0$ gives 6; 4×2 is 8; $+3$ is 11; no remainder.

Note that first the number is found which added to $58 (2 \times 29)$ will give 76; this is 18, first remainder. Secondly, the number is found which added to 6 times 29 $(6 \times 9 + 6 \times 20)$ will give 185; this gives 11 as second remainder. Finally 4×29 (as 4×9 and 4×20) exactly equals 116.



Chap. V. Division. 55

I shall not, I (hope) need to trouble myself, or Learner, to shew the Working of this Sum, or any other, having, now (as I suppose) sufficiently treated of Division; but will leave it to the Censure of the experienc'd to judge, whether this Manner of dividing be not plain, lineal, & to be wrought with fewer Figures than any which is commonly taught: As for **Example** *appeareth,*

```
                 (8
                97 (5
              9863 (0
            987529 (3
          98764181 (0
        9876522609 (8
      987654195987 (6
    493827148548765 (4
  2469135786376543 (2
 1234567899876543 21          (124999999
 9876543211111111             987654321
  9876543222222222            124999999
   98765433333333             2499999982
    987654444444              3749999974
     98765555 5               4999999966
      9876666                 6249999958
       98777                  7499999940
        988                   8749999933
         9                    9999999920
                              11249999915
                                   8
                  Proof 12345678998765432I
```

E 2 CHAP.

HODDER'S ARITHMETIC, BOSTON, 1719
Long division with check
$12345678998765 4321 \div 987654321$

Other variations. Many of the older arithmetics recommend writing first in a column the first multiples of the divisor, for use in division. Occasionally the printed form gives all the remainders in a vertical column, not preserving the decimal order; this is probably frequently the printer's error. Cocker's *Decimal Arithmetic* shows this peculiarity in several division examples. Another peculiarity which connects with the older process of division consists in repeating the divisor under the proper place in the dividend, and then under it the product by the corresponding digit of the

quotient. Humphrey Baker teaches a modification of this procedure, subtracting above, but both Baker and Recorde constantly employ the scratch method.

For several centuries one who could perform long division was considered an expert mathematician. Today in oriental countries, even in Arabia and in India, one would with difficulty find a native who has this ability.

Bibliography for Supplementary Reading

TROPFKE, *Geschichte der Elementar-Mathematik in systematischer Darstellung*. Vol. I, *Rechnen*. Second ed. Leipzig, 1921.

This work in German takes up the fundamental operations and other topics of arithmetic systematically in the historical development. There is no work in English which corresponds in completeness to this work.

L. L. JACKSON, "The educational significance of sixteenth-century arithmetic from the point of view of the present time." New York. Teachers College, Columbia University, 1906.

SUSAN R. BENEDICT, "A Comparative Study of the Early Treatises introducing into Europe the Hindu Art of Reckoning." Thesis, University of Michigan. Privately published, 1916.

L. C. KARPINSKI, "Two Twelfth-Century Algorisms." *Isis*, Vol. III, pp. 396–413, 1921.

FLORIAN CAJORI, *History of Elementary Mathematics*. New York. Macmillan.

D. E. SMITH, *History of Mathematics*. Boston. Ginn & Co., 1921. Vol. I, 1923. Vol. II, promised for early publication, will furnish desirable additional material on arithmetic.

D. E. SMITH'S *Rara Arithmetica* (Boston, 1908) contains many illustrations showing the fundamental operations as given in the early texts.

In almost every large library some of the early arithmetics mentioned in this and the preceding chapter may be found. The examination of an original copy of this kind is an interesting and profitable exercise for any teacher of arithmetic.

CHAPTER V

FRACTIONS

Common Fractions

Multiplication is vexation,
Division is as bad;
The Rule of Three perplexes me,
And Fractions drive me mad.

Egyptian and Babylonian fractions. Fractions have always occasioned difficulty for teachers of the art of arithmetic. A fundamental part of the early Egyptian arithmetic consists in the explanation of operations with fractions. With the contemporary student of arithmetic in Babylon, fractions also constituted an important part of the arithmetic. Both types of treatment profoundly influenced arithmetic for three thousand years, and the Babylonian peculiarities confront us hourly, whenever we note the time of day.

The concept of a numerator and a denominator, combined in a particular way to form a fraction, is obviously complicated. The Egyptians sought to escape the difficulty by confining their attention to unit fractions, having the numerator unity, with the single exception of two-thirds. This required the writing of long series of fractions. Thus seven-eighths was written as $1/2$, $1/4$, $1/8$ or as $2/3$, $1/6$, $1/12$. The illustration reveals one weakness of this system, namely, that the separation is not unique. The earliest numerical table which has come down to us consists of the Egyptian conversion into unit fractions of fractions with numerator 2 and odd denominators from

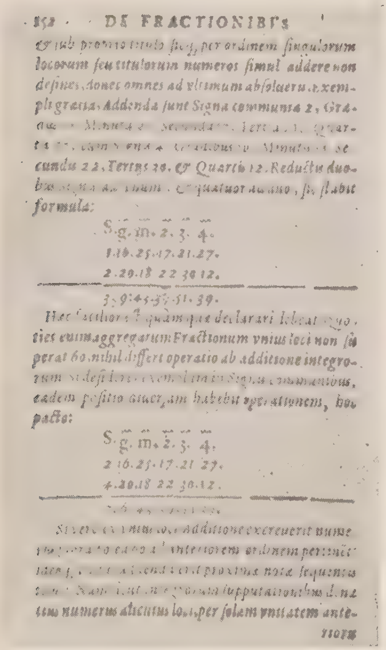
5 to 99. Herein $\frac{2}{5}$ is given as $\frac{1}{3}$, $\frac{1}{15}$; $\frac{2}{15}$ as $\frac{1}{10}$, $\frac{1}{30}$, and so on, occasionally employing 4 or 5 unit fractions. These Egyptian fractions occupied a large place in later Greek arithmetic, wherein also an exception is made in a special symbol and special treatment for $\frac{2}{3}$. Egyptian problems on fractions appear in Greek papyri from the fourth to the ninth centuries of the Christian Era.

The Ahmes papyrus contains problems concerning the division of loaves of bread, not exceeding nine, among ten people. The results are expressed in unit fractions. The Greek papyri, many centuries later, have the same problems couched in abstract terms.

Minutes and seconds. The Babylonian sought escape from fractional difficulties by confining his attention to sexagesimal fractions, that is, with denominators 60 and powers of 60. This is precisely our present method with decimal fractions, using however the denominator 10. In particular the Babylonians applied these fractions to the circle and to the measurement of time; from this they progressed to the application of these subdivisions to weights and other measures. This logical and psychological advance with decimal subdivisions is today being advocated by many scholars and manufacturers. The burden which our awkward system of weights and measures places upon our school children is tremendous; the bad effect upon our export trade is serious. Teachers should be conscious of the great advantages offered by the metric system.

Babylonian influence on Greek astronomy. The Babylonian astronomy exerted from early times great influence upon the Greek astronomy. In the second century B.C.

Hipparchus, father of astronomy, introduced Babylonian fractions into Greek astronomy. From that time to this they have remained in astronomical computations. During the Middle Ages these fractions were applied to


152 DE FRACTIONIBVS

& sub proximo titulo [illegible], per ordinem singulorum locorum seu titulorum numeros simul addere non desines, donec omnes ad ultimum absolueris. exempli gratia, Addenda sunt Signa communia 2, Gradus [illegible] Quartis [illegible] secundis 22, Tertiis 30, & Quartis 12. Reductis duobus [illegible] quatuor [illegible], sic stabit formula:

S. g. m. 2. 3. 4.
1. 16. 25. 17. 21. 27.
2. 20. 18 22 30 12.

3. 39. 43. 39. 51. 39.

Hæc [illegible] quàm quæ declarari debeat [illegible] ties cum aggregatum Fractionum unius loci non superat 60, nihil differt operatio ab additione integrorum [illegible] Signis communibus, eadem positio [illegible] habebis operationem, hoc pacto:

S. g. m. 2. 3. 4.
2 16. 25. 17. 21 27.
4. 20. 18 22 30. 12.

[illegible]

[illegible] additione excreuerit nume[illegible] inferiorem ordinem pertinet [illegible] proximus nota sequentis [illegible] supputationibus [illegible] tus numerus alicuius loci per solam unitatem anteriorem

GEMMA FRISIUS, COLOGNE, 1576

Treatment of "astronomical" fractions in the arithmetic of Gemma Frisius (*see* page 69).

The various subdivisions are signs of the zodiac (S.), 30°, degrees (g.), minutes (m.), seconds (2), thirds or $\frac{1}{60^3}$ (3), and fourths or $\frac{1}{60^4}$ (4).

In the first addition problem we have:

S.	g.	m.	2.	3.	4.
1	16	25	17	21	27
2	20	18	22	30	12
3	39	43	39	51	39

wherein the author suggests the use of a "sign" of 60°. In the problem below the ordinary sign of 30° is used.

S.	g.	m.	2.	3.	4.
2	16	25	17	21	27
4	20	18	22	30	12
7	6	43	39	51	39

In the second column the 30° have made one sign to be carried. Such explanations were common up to 1650 A.D.

all computations, replacing in large measure the unit fractions of Egypt. The designation *astronomical* or *physical* fractions was applied to them. Leonard of Pisa expresses the approximate root of a third-degree equation in sexagesimal fractions carried to the eighth place. This corresponds to our solution by Horner's method to the twelfth place, using decimal fractions. Our words

"minutes" and "seconds" go back to Latin forms *minutiae primae*, *minutiae secundae*, meaning first fractions, second fractions, and so on.

Roman fractions; apothecary tables. The Romans simplified the fractions following the Babylonian pattern. The base was chosen as 12, which appears also to be an original Babylonian subdivision. This was applied in Rome first to the unit of weight, the *as;* the twelfth of this *as* was the *uncia*, from which we get our words "ounce" and "inch." Note that again after beginning with the concrete, the fractional numbers used are made abstract to apply to other measurements.

Roman fractions had a further complication in that special symbols were devised and used for $1/12$ to $11/12$, for $1/8$ as one and one-half twelfths, $1/24$, $1/36$, $1/48$, $1/96$, $1/144$, and on to $1/576$ and to even smaller fractions. In spite of the great difficulty of operating with these symbols, they continued in arithmetical instruction during the tenth to the thirteenth centuries. Our present apothecary weight symbols trace back to these Roman devices. Upon the Roman abacus separate little bars were provided for certain of the more common fractions.

Greek fractions. The Greeks employed common fractions as well as unit and sexagesimal, writing the numerator with one accent mark and the denominator written twice with two accent marks. Unit fractions were indicated simply by the denominator with one accent, quite similar to the Egyptian procedure, which was to write the denominator surmounted by a heavy dot. Occasionally the Greeks wrote the numerator with the denominator in the position where we write an exponent.

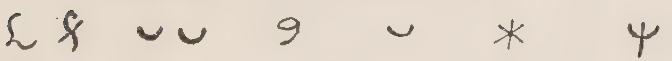
As	Deunx	Decunx vel Dextans	Dodrans	Bisse	Septunx	Semis	Quincunx
×	ϛϛϛ	ϛϛϛ	ϛϛ	ϛϛ	ϛ	ϛ	ϛϛ
~	~	~	~	~	~	~	~
XII	XI	X	VIIII	VIII	VII	VI	V
ſſ	ſſ	ſſ	ſſ	ſſ	ſſ	ſſ	ſſ
CC	CC	CC	CC	C	C	C	C
LXXX	LX	XL	XVI	XC	IX	XL	XX
VIII	IIII			II	VIII	IIII	
υ	υ	υ	υ	υ	υ	υ	υ
LXXII	LXVI	LX	LIIII	XLVIII	XLII	XXXVI	XXX

ROMAN FRACTIONS FROM A PRINTED COPY (PARIS, 1867) OF THE WORKS OF POPE SYLVESTER II (GERBERT, *c.* 1000 A.D.), EDITED BY OLLERIS

The Roman *as* with its duodecimal subdivisions. The first column reads down, giving the *as* as equal to *unciae 12* or *scripuli 288* or *sextulae 72.*

Triens	Quadrans	Sextans	Sescuncia	Uncia
ϛϛ	ϛ	ϛ	£	~
~	~	~	~	ſſ
IIII	III	II	I ϛ	XX
				IIII
ſſ	ſſ	ſſ	ſſ	OBL
XC	LXX	XL	XXX	XL
VI	II	VIII	VI	VIII
				uncia
				Γo solidi
0	0	0	0	υ
XXIIII	XVIII	XII	VIIII	VI

Continuation of above table, giving $\frac{4}{12}$, $\frac{3}{12}$, $\frac{2}{12}$, $\frac{1}{8}$, and $\frac{1}{12}$ in lower units.



$\frac{1}{24}$ $\frac{1}{24}$ semuncia; $\frac{1}{36}$ duella; $\frac{1}{48}$ sicilicus; $\frac{1}{72}$ sextula; $\frac{1}{96}$ dragma; $\frac{1}{144}$ hemisecla

$\frac{1}{216}$ tremissis; $\frac{1}{288}$ scripulus; $\frac{1}{576}$ obolus; $\frac{1}{1152}$ cerates; $\frac{1}{1728}$ siliqua; $\frac{1}{2304}$ chalcus

SMALLER SUBDIVISIONS OF THE "UNCIA"

Hindu and Arabic fractions. The Hindu fractional forms were similar to the form which we use, without the bar. Mixed numbers were written with the integral part above the fraction; thus $8\frac{3}{11}$ was written $\begin{matrix}8\\3\\11\end{matrix}$. Brahmagupta gave systematic rules for the fundamental operations with fractions. Extensive treatment, including negative forms, is given by Mahavir and also by Bháskara. The latter writer says: "After reversing the numerator and denominator of the divisor, the remaining process for division of fractions is that of multiplication"; Brahmagupta reduces dividend and divisor to a common denominator, before inverting; and Aryabhata indicates the same procedure.

The complications of unit fractions, common fractions, and sexagesimal fractions were augmented by an Arabic device which fortunately made little impression on European writers beyond Leonard of Pisa. This Arabic device consisted in writing a fractional form $\frac{1}{3}\ \frac{1}{5}$ to mean $\frac{1}{3}+\frac{1}{3}$ of $\frac{1}{5}$ or $\frac{4}{13}\ \frac{3}{11}$ to mean $\frac{4}{13}+\frac{3}{11}$ of $\frac{1}{13}$; an elaborate treatment of such forms was given by Al-Hassar, probably in the twelfth century, translated into Hebrew by Moses ben Tibbon in the thirteenth.

The word "fraction." Our notation of fractions is quite certainly based upon Arabic forms without the bar, these being derived from the Hindu. The Arabic word for fraction, *al-kasr*, is derived from the stem of the verb, meaning "to break." The early writers on algorism commonly used *fractio*, while Leonard of Pisa and John of Meurs (fourteenth century) use both *fractio* and *minutum ruptus* or *ruptus*. Early writers in English

frequently used the corresponding expression, "broken numbers." The two earliest English algorisms, mentioned above, do not contain any discussion of fractions, except incidentally one-half; in this these manuscripts follow Sacrobosco and Alexandre de Ville Dieu.

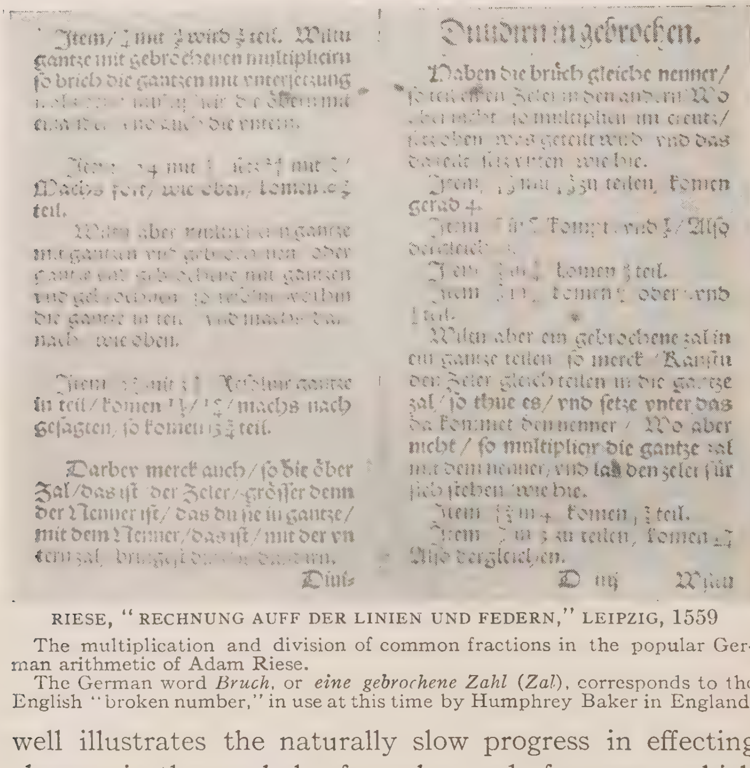
Common or vulgar fractions. The modern treatment and the terminology of common fractions appears in Recorde's *Grounde of Artes*, with the exception only of the expression "common" fraction or "vulgar" fraction. This latter designation was used after the introduction of decimal fractions to distinguish the ordinary from the decimal fractions. Continental writers, like Peurbach of the fifteenth century in his arithmetic printed in 1534 and earlier, used the Latin expression *fractiones vulgares* or *minutiae vulgares* to distinguish these from the sexagesimal fractions. In English the word "fraction" appears to have been used first by Chaucer (1321).

In early American arithmetics the designation "broken numbers" was used as an alternate for "fractions." "Vulgar" was applied to common fractions to distinguish them from "decimal fractions" or "decimals" (Pike, 1788); the treatment involved few modifications from present procedure or terminology.

Decimal Fractions

The forerunners of decimal fractions. A thousand years intervened between the discovery of the simple device for representing all integers in a decimal scale by nine symbols with a zero and the extension of the same principle to fractions in a decimal scale downward. Numerous approaches to the fundamental principles of

decimal fractions were made. The summary of these steps leading to the development of decimal fractions



Dividirn in gebrochen.

Haben die brüch gleiche nenner/

Darber merck auch / so die öber Zal / das ist der Zeler / grösser denn der Nenner ist / das du sie in gantze / mit dem Nenner / das ist / mit der vnternzal [illegible]

Wilt aber ein gebrochene zal in ein gantze teilen [illegible]

RIESE, "RECHNUNG AUFF DER LINIEN UND FEDERN," LEIPZIG, 1559

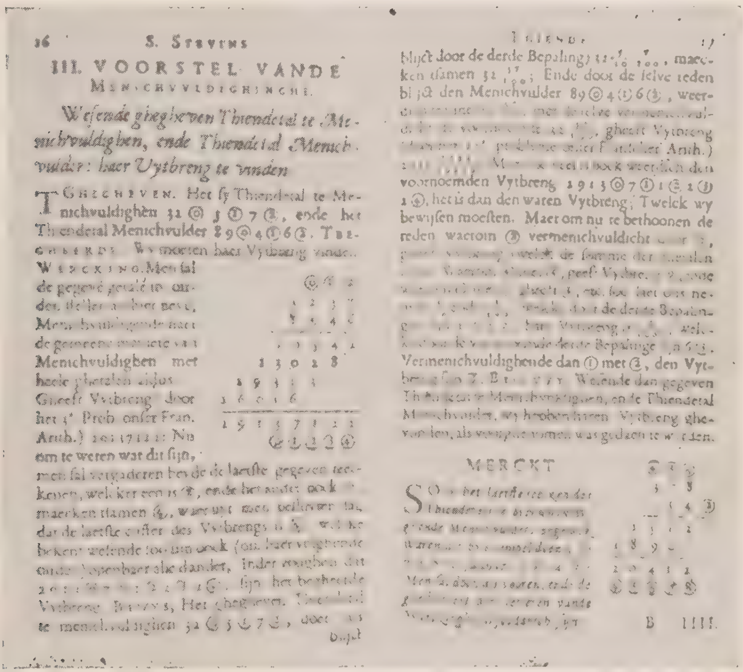
The multiplication and division of common fractions in the popular German arithmetic of Adam Riese.

The German word *Bruch*, or *eine gebrochene Zahl* (*Zal*), corresponds to the English "broken number," in use at this time by Humphrey Baker in England.

well illustrates the naturally slow progress in effecting changes in the symbols of number and of measure which the workaday world employs.

The Babylonian system of fractions, to which we owe our minutes and seconds, corresponds in the scale of 60 to decimal fractions in the scale of 10. The same symbolism of primes was later used by some writers for decimal fractions. In approximations of square root and cube

root results were frequently given in sexagesimal fractions. However, Johannis de Muris in the fourteenth



16 S. STEVINS

III. VOORSTEL VANDE MENICHVULDIGHINGHE.

Wesende ghegheven Thiendetal te Menichvuldighen, ende Thiendetal Menichvulder: haer Vytbreng te vinden

TGHEGHEVEN. Het sy Thiendetal te Menichvuldighen 32 ⓪ 5 ① 7 ②, ende het Thiendetal Menichvulder 89 ⓪ 4 ① 6 ②.

blijckt door de derde Bepaling; 32 $\frac{5}{10}$ $\frac{7}{100}$, maecken tsamen 32 $\frac{57}{100}$; Ende door de selve reden blijckt den Menichvulder 89 ⓪ 4 ① 6 ②, weert

voornoemden Vytbreng 2913 ⓪ 7 ① 1 ② 2 ③ 2 ④, het is dan den waren Vytbreng; Twelck wy bewijsen moesten.

MERCKT

SIMON STEVIN, "LA THIENDE," LEYDEN, 1585

First printed work to give a discussion of decimal fractions, appearing both in Flemish and in French (*Disme*) in 1585.

At the top of the right-hand page in the third line 89.46 is indicated by 89 followed by 0 within a circle, 4 followed by 1 within a circle, and 6 followed by 2 within a circle. Incidentally it may be mentioned that this same notation was used for algebraic purposes by Stevin; with two plus signs interjected this expression just given would represent $89+4x+6x^2$.

The decimal point appears to have been used first in 1616 in the works of John Napier, inventor of logarithms.

century gave the square root of 2 as 1·4·1·4, saying that the 1 represented units, the first 4 tenths, the second 1 "tenths of tenths," and the second 4 "tenths of tenths of

tenths." However, he then extended this, writing the result also to twentieths of twentieths of twentieths, finally giving the result in sexagesimal fractions.

The square root of 2 was obtained by writing 2,000,000, extracting the root of this, dividing by 1000, and reducing the result to sexagesimal fractions. This method appears in manuscripts of the twelfth century and in printed books of the sixteenth century.

Special rules for division by integral multiples of 10, 100, and the like appeared before the invention of printing. This led to an actual decimal point in one problem in the treatise by Pellizzati (1492) but the author makes no further use of the device.

Interest problems brought Christian Rudolff to a practical use of decimal fractions in computing compound interest. His mark of separation is a vertical bar, which was quite frequently used by later writers who gave an explanation of decimal fractions.

The trigonometric functions had a large part in emphasizing the practical necessity of some simple device for extended computations. The Greeks gave a table of chords in a circle with radius 60; for any refined computations this required the use of primes and seconds or more. Peurbach about the middle of the fifteenth century determined to use in a table of sines the radius 60,000 or 600,000; his able pupil Regiomontanus extended this to 6,000,000, and finally to 10,000,000. The Hindus and the Arabs gave the shadow function, or cotangent, with a stick of length 12. Regiomontanus here also adopted a decimal base 100,000. In both cases only a decimal point was necessary with a unit radius to give the modern tables.

Simon Stevin discovers decimal fractions. The first systematic discussion of decimal fractions with full appreciation of their significance was given by Simon Stevin of Bruges in 1585. His work in Flemish, entitled *La Thiende*, was published at Leyden by the famous Plantin press. This was republished again in 1585 in French with the title *La Disme;* in 1608 an English translation by Robert Norton, *The Art of Tenths or Decimall Arithmetike*, appeared in London.

This work is addressed to astronomers, surveyors, masters of money (of the mint), and to all merchants. Stevin says, of this work, that it treats of "something so simple, that it hardly merits the name of invention." He adds:

"We will speak freely of the great utility of this invention; I say great, much greater than I judge any of you will suspect, and this without at all exalting my own opinion . . . . For the astronomer knows the difficult multiplications and divisions which proceed from the progression with degrees, minutes, seconds and thirds . . . . the surveyor, he will recognize the great benefit which the world would receive from this science, to avoid . . . . the tiresome multiplications in Verges, feet and often inches, which are notably awkward, and often the cause of error. The same of the masters of the mint, merchants, and others . . . . But the more that these things mentioned are worth while, and the ways to achieve them more laborious, the greater still is this discovery *disme*, which removes all these difficulties. But how? It teaches (to tell much in one word) to compute easily, without fractions, all computations which are encountered in the affairs of human beings, in such a way that the four principles of arithmetic which are called addition, subtraction, multiplication and division, are able to achieve this end, causing also similar facility to those who use the casting-board (*jetons*). Now if by this means will be gained precious time; . . . . if by this means labor, annoyance, error, damage, and other accidents commonly joined with these

DISME:

The Art of Tenths,

OR,

Decimall Arithmetike,

Teaching how to performe all Computations *whatsoeuer, by whole Numbers without* Fractions, by the foure Principles of *Common Arithmeticke: namely,* Addition, Substraction, Multiplication, and Diuision.

Inuented by the excellent Mathematician, Simon Steuin.

Published in English with some additions by *Robert Norton*, Gent.

Imprinted at London by *S. S.* for *Hugh Astley*, and are to be sold at his shop at Saint Magnus corner. 1608.

ENGLISH TRANSLATION OF THE FIRST WORK ON DECIMAL FRACTIONS

The notation used in this text is the same as that employed in *La Disme* by Stevin in 1585. From *disme* we have the word "dime."

computations, be avoided, then I submit this plan voluntarily to your judgment."[1]

What can one add to these words of the first writer on the subject, and an independent discoverer of decimal fractions? All that Stevin says applies today, hardly with the change of a letter. The genius of Stevin is evident in the comprehensive grasp which he had of the universal application of decimal fractions to affairs. Much of the benefit of this invention is lost to us in America, because we persist in using non-decimal weights and measures.

Evolution of the decimal point. The symbolism of Stevin consisted in marking the place of units by a zero within a circle, and each decimal place by the digit corresponding to the number of the decimal place inclosed within a circle, following or above or below the digit of the numerator of the decimal fraction to which it appertained. Such an awkward notation could not survive, for even the author tried three variations of this symbolism on one page. The transition to the decimal point as now used in America and England, or the comma as used on the Continent, was independently effected by several early writers.

The immediate application of decimal fractions was made particularly to the trigonometric functions and to logarithms. The decimal point in print appears in 1616 in the English translation of Napier's fundamental work on logarithms. Many writers to the end of the seventeenth century used awkward notations; thus Milliet de Chales, in his encyclopedic work on the mathematical

[1]Simon Stevin, *La Disme*, 1585.

Vulgar and *Decimal.* 49

NOTATION.

The TABLE.

Whole Numbers							Decimal Parts.						
	6	5	4	3	2	1	, 1	2	3	4	5	6	
&c.	*Hundred of Thousands.*	*Tens of Thousands.*	*Thousands.*	*Hundreds.*	*Tens.*	*Units Place.*	*Parts of Ten, or* $\frac{1}{10}$.	*Parts of a Hundred*	*Parts of a Thousand*	*Parts of Ten Thousand*	*Parts of a Hundred Thousand*	*Parts of a Million*	&c.

The USE.

Of this *Table* will appear in the following *Observations.*

I.

That *Decimal Fractions* are always seperated from *whole Numbers* by some distinguishing Mark, as a *Comma*, a *Period*, or the like. So 654321, are *Integers*; and ,123456 *Decimal Parts.* And from hence is derived a *Universal Rule* to distinguish *Integers* from *Decimals*, in any *mix'd Sums* whatsoever, viz. *That the Integers always lay on the left, and the Decimals on the Right Hand of the Separatrix.*

II.

The *Denominator* is always omitted in the *Notation* of *Decimal Fractions*; Thus, ,1 is the *Notation* of $\frac{1}{10}$.

G 2 DECI-

DECIMAL FRACTIONS AS PRESENTED BY GREENWOOD IN 1729

sciences, published in 1690 after the death of the author, used the left half of a pair of brackets. Cavalerius in his *Trigonometria* (Bologna, 1643) uses the decimal point and gives a full explanation of the subject. During the seventeenth century the arithmetics which avoided any mention of decimal fractions were about as numerous as those which gave some treatment of them.

Advantages of decimal fractions. By the eighteenth century the utility of these geometrical fractions, as they were sometimes termed, had been demonstrated so often and so clearly that the treatment of this subject became a regular part of arithmetic. English texts of the early eighteenth century commonly treated the decimal arithmetic extensively.

The American texts of the eighteenth century included full discussion of decimals, using the word *separatrix* to designate the decimal point. Isaac Greenwood in his *Arithmetick Vulgar and Decimal* of 1729, and Nicholas Pike in 1788 not only give a modern treatment of the subject, but both include the abbreviated process to obtain the product to any given number of places by reversing the multiplier and the abbreviated process in division. To such complete explanations of decimal notation is undoubtedly due the adoption of decimal coinage in 1785 by the Continental Congress.

The development of decimal fractions illustrates the process of evolution in the realm of mathematical ideas. As we trace the steps culminating in this useful device it is perfectly evident that a succession of thinkers made possible this attainment. Similarly in practically every advance in arithmetic, algebra, and trigonometry, as in

all science, a host of intellectual workers have participated to make the advance possible. In the field of science we are truly the heirs of all the ages past.

Bibliography for Supplementary Reading

Sir William Ridgeway, *Roman Measures and Weights*, and *Roman Money*, in Sandys, "A Companion to Latin Studies," 3d ed., Cambridge University Press, 1921.

Aubrey Drury *et al.*, *World Metric Standardization*. San Francisco, 1922.

L. L. Jackson, *loc. cit.*, pp. 85–110.

See the references to Chapter I.

Consult the encyclopedias and dictionaries under Decimal, Fraction, Minute, Second, Time (measurement of), and Trigonometry (angles).

CHAPTER VI

BUSINESS ARITHMETIC

Applied arithmetic. The application of arithmetic to commercial problems extends from earliest times of historical record to the present day. However, among the Greeks and in Europe until towards the end of the fourteenth century these applications did not become the material for written exposition. The early European algorisms present simply the technique of arithmetic, without any practical applications. The single exception to the rule is the work of Leonard of Pisa (1202 A.D.) which directly under Arabic influence gave a sufficiently extensive treatment of arithmetic to permit inclusion of problems on business.

Everyday problems of the Egyptians. The Egyptian arithmetic might well be taken as a model today in that the problems of that ancient day are concerned with the daily life of the people. How much corn does it take to stuff a goose? How many sacks of flour in a given granary? What is the cost of manufacturing a certain fine piece of jewelry? What is the cost of making 20 gallons of beer? For the preceding, the formula is given.

The pedagogical soundness of Egyptian procedure was appreciated by Plato, who said (*Laws*, 819):

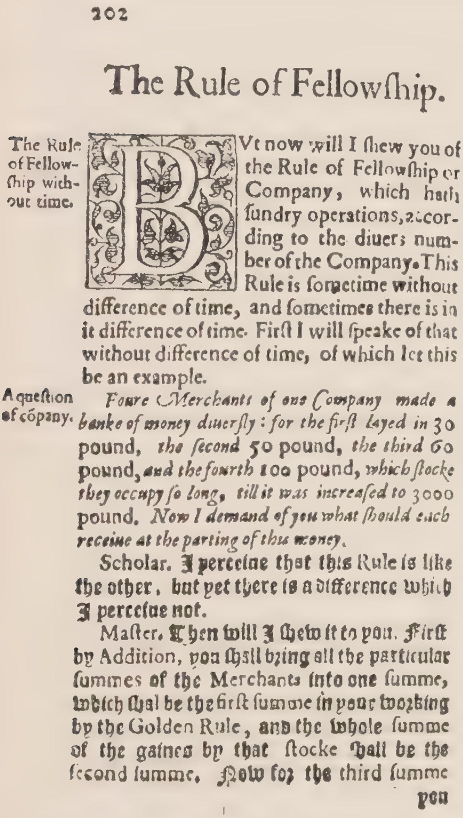
"All freemen, I conceive, should learn as much of these various disciplines as every child in Egypt is taught when he learns his alphabet. In that country, systems of calculation have been actually invented for the use of children, which they learn as a pleasure and amusement. They have to distribute apples and garlands, adapting the same number either to a larger or less number of persons. . . .

Another mode of amusing them is by taking vessels of gold, and brass, and silver, and the like, and mingling them or distributing them without mingling; as I was saying, they adapt to their amusement the numbers in common use, and in this way make more intelligible to their pupils the arrangements and movements of armies and expeditions, and in the management of a household they make people more useful to themselves, and more wide awake; and again in measurements of things which have length, and breadth, and depth, they free us from that ludicrous and disgraceful ignorance of all things which is natural to man."

Practical arithmetic of the Hindus. The Hindus, no less than the more ancient Egyptians, had a fondness for arithmetic as applied to the world of affairs. Brahmagupta, following Aryabhata, gives the Rule of Three, applied to "the barter of commodities," employing a terminology later adopted in translation by the Arabs. Principal and interest, partnership, and gain in trade also make their initial appearance in systematic form in India with Aryabhata and Brahmagupta, and continue in prominence in the Hindu treatises of Mahavir, Sridhara, and Bháskara. Mensuration as an arithmetical exercise is also a favorite Hindu topic, but this appeared earlier in Greece in the works of Heron of Alexandria (*c.* 50 A.D.). The work on arithmetical and other series is more detailed in India than in Greece, following algebraic lines.

Arabic business arithmetic. The practical work of the Hindus was utilized by several Arabic writers, and particularly the work on series. A systematic exposition of Arabic commercial arithmetic was written by Abu 'l Wefạ (940–998 A.D.) of Bagdad, one of the most famous astronomers of his day. His arithmetic includes such topics as duties, exchange, bookkeeping, commercial

operations, mensuration, and weights and measures. The Hindu Rule of Three is found generally in Arabic alge-



202

The Rule of Fellowſhip.

The Rule of Fellowſhip without time.

BVt now will I ſhew you of the Rule of Fellowſhip or Company, which hath ſundry operations, according to the diuers number of the Company. This Rule is ſometime without difference of time, and ſometimes there is in it difference of time. Firſt I will ſpeake of that without difference of time, of which let this be an example.

A queſtion of cōpany. *Foure Merchants of one Company made a banke of money diuerſly: for the firſt layed in* 30 pound, *the ſecond* 50 pound, *the third* 60 pound, *and the fourth* 100 pound, *which ſtocke they occupy ſo long, till it was increaſed to* 3000 pound. *Now I demand of you what ſhould each receiue at the parting of this money.*

Scholar. I perceiue that this Rule is like the other, but yet there is a difference which I perceiue not.

Maſter. Then will I ſhew it to you. Firſt by Addition, you ſhall bring all the particular ſummes of the Merchants into one ſumme, which ſhal be the firſt ſumme in your working by the Golden Rule, and the whole ſumme of the gaines by that ſtocke ſhall be the ſecond ſumme. Now for the third ſumme

you

PARTNERSHIP AS PRESENTED IN RECORDE'S "GROUNDE OF ARTES," LONDON, *c.* 1542. (*Illustration from a later edition*)

The rule of fellowship was a continuation in English of the Italian problems of a similar nature. The Italian commercial arithmetics of the late fifteenth and the sixteenth centuries greatly influenced European and British arithmetic. In Italy this type of work is found first in the treatise of Leonard of Pisa, written in 1202 A.D.

braic works, particularly in Al-Khowarizmi's algebra (*c.* 825 A.D.), in Al-Karkhi's arithmetic (*c.* 1010 A.D), in the arithmetic by Al-Kalasadi, a Spanish Arab of the fifteenth century, and in the sixteenth-century work by Beha Eddin (1547–1622).

Italian commercial problems. Greek and Roman use of the principles of business arithmetic are indicated by numerous early references to the subjects of interest, inheritance, and mixture or alligation. Leonard of Pisa,

THE

CONTENTS.

FIRST TWO PAGES OF THE TABLE OF CONTENTS OF GREENWOOD'S "ARITHMETICK," BOSTON, 1729

the first European writer on commercial arithmetic, devotes much more space to the applications than to the theory of arithmetic. His treatise contains numerous Hindu and Arabic problems, a few Greek and Roman, and one or two which are apparently Chinese; all of this

material is systematically expounded by the great Italian who demonstrated repeatedly his own genius in mathematical fields. The illustrations on the prices of merchandise, on the exchange of moneys, on partnership, on

CONTENTS.

CONTENTS OF GREENWOOD'S "ARITHMETICK," 1729

The topics of business arithmetic are indicated in the headings.

compound proportion, on the rule of false position, on capital and gain, and on miscellaneous problems constituted for centuries the storehouse from which other writers drew material. The words "capital" (Leonard of Pisa), "percent" and the symbol, "debit," and "credit" are all original with Italian writers, the final

two words being found in Paciuolo's *Summa d'Arithmetica* of 1494, which includes the first treatise on bookkeeping.

Italy was a center of trade from the thirteenth to the sixteenth centuries. Leonard of Pisa's father was an agent, a *factor*, in charge of a distribution center, a *factorie*, in northern Africa. There it was that Leonard learned the new numerals "instructed by a grocer." It is interesting to note that American arithmetics as late as 1870 used the word "factor" as meaning "agent."

In consequence of their commerce, Italians were for centuries particularly interested in arithmetic. Manuscript treatises of the fourteenth century give numerous problems on equation of payments and other topics of commercial arithmetic.

The Treviso arithmetic of 1478, the widely popular and oft reprinted work by Borghi entitled "la nobel opera de arithmethica" of 1484, Calandri's illustrated arithmetic of 1491, Pellizzati's *Art de arithmeticha* of 1492, and Paciuolo's *Summa* of 1494 are the great commercial arithmetics published in Italy during the fifteenth century (*incunabula*); only one other commercial arithmetic was published during this period, Widman's *Behennd und hüpsch Rechnung uff allen Kauffmanschafften*, Leipzig, 1489.

Partnership with time, barter, interest, alligation, and a host of other topics taught in many continental schools and in American schools even in the twentieth century are included in these early commercial arithmetics. Such words as merchant, company, tariff, duty, payment, as well as debit, credit, percent, factor, and capital, are directly traceable to the popularity of the Italian arithmetics. American arithmetic exhibits today the influence

of the Italian civilization of the fifteenth century, which contributed so largely to the discovery and early explora-

Chap. XVII. 137

CHAP. XVII.

A most brief & compendious way of working all manner of Questions of Interest upon Interest

Example.

First, State your Question thus:

If 100 *l.* gain 6 *l.* what the Principal?

2. Multiply the second and third Numbers together, and divide by your first, which is done by cutting off two first Figures of the Pounds with a line.

3. Multiply them by 20, by 12, and 4, and all above 2 figures in each Multiplication carry over the line unto the left, as you see in these following Examples.

If 100 *l.* in 12 Months gain 6 *l.* what will 356 *l.* gain in 18 Months?

If 100 *l.*——6 *l.*——356 *l.*

6

21|36

20

	l.	*s.*	*d.*
12 *Mon. fa.*	21	07	2 ¼
6 *Mon. fa.*	10	13	7
	32	00	9 ¼

7|20

12

2|40

4

1|60

K 3 275 *l.*

COMPOUND INTEREST AS TREATED IN THE COLONIAL ARITHMETIC OF HODDER, 1719

The multiplication by 20 gives shillings, by 12 gives pence, and by 4 gives farthings. Note that the interest on 356 pounds for 1 year at 6% is 21 £ 7s. 2¼d.

The fa. is abbreviation for the Latin *facit*, "it makes."

tion of the New World which now bears an Italian's name. Amerigo Vespucci made his first journeys to Spain while engaged in Italian commercial enterprises. It was not only the Italian commerce but also the interest in navigation and discovery which proved a fine stimulus to the study of mathematics.

Recorde's list of applied topics is largely self-explanatory: the Golden Rule, or Rule of Proportion direct, called

the Rule of Three; the Golden Rule Reverse, and Double, and Compound; the Rule of Fellowship; the Rule of Alligation; and the Rule of Falsehood. Humphrey Baker includes, as do the seventeenth-century editions of Recorde, exchange and weights and measures as well as the topics mentioned above.

American commercial arithmetic. Our American Pike (1789) apparently determined to include all possible applications of arithmetic. The Table of Contents covers six pages, touching more than one hundred separate arithmetical topics, as well as numerous others. The inclusion of annuities and of the tables of the compound interest functions is particularly worthy of note, as these functions are now returning to the American college texts on freshman mathematics.

The reason why the early arithmetics contained such complete discussions of commercial arithmetic lies undoubtedly in the fact that this topic was studied largely by adults, and not by children. In American schools until long after the Civil War, and in rural schools of the nineteenth century, pupils of the seventh and eighth grades were mature; the great majority passed from the eighth grade into active business life. At the present time even in rural schools children of twelve to fourteen years of age are found in the seventh and eighth grades. For these children elementary algebra and constructive geometry are much more suitable than commercial arithmetic, which might well be presented in the tenth and eleventh grades, when the pupils are mature enough to be interested in topics which relate directly to business. The complexities of modern business arithmetic are greater

than those of elementary algebra and constructive geometry, which subjects are now found in seventh- and eighth-grade textbooks replacing the material which represented bygone days and conditions.

Bibliography for Supplementary Reading

Clive Day, *A History of Commerce.* Rev. ed. New York, Longmans Green & Co., 1922.

C. A. Herrick, *History of Commerce and Industry.* New York, Macmillan Co., 1917.

L. L. Jackson, *The Educational Significance of Sixteenth Century Arithmetic.* New York, 1906.

F. Cajori, *A History of Elementary Mathematics.* Rev. ed. New York, 1917.

D. E. Smith, *History of Mathematics.* Ginn and Co., 1923.

A rapid survey of the history of Europe from 1000 A.D. to 1600 A.D., as found in any General History, will give to the teacher the necessary background for the work of this chapter. At the same time the development of the arithmetic will illuminate the general history of the period.

Consult also encyclopedias under Commerce.

CHAPTER VII

THE TERMINOLOGY OF ARITHMETIC

The progress of arithmetic. The words used in arithmetic reflect in some measure the historical progress of arithmetical science. The Egyptians, the Babylonians (indirectly), the Greeks, the Romans, the Europeans of the Middle Ages, the Normans and the Anglo-Saxons, the French, the Hindus and the Arabs, and even the Americans are represented. To assign each word to its proper place is no easy task; indeed, in several instances authorities would disagree as to whether a given word entered, for example, through the French or through the Latin. However, the purpose in this chapter is to present in broad outline the fundamental facts concerning the terminology found in American arithmetics, with indication of the genesis and development of the terms.

Greek and Roman influence. The Greek influence is seen in words relating to mensuration, due to the Greek devotion to geometry. To the Greeks we owe the separation of mathematics into the four great fields of arithmetic, geometry, astronomy, and music. This philosophical tendency of the Greeks is reflected in the fact that the words mentioned, and the word "mathematics" as well, are Greek.

The Latin words in arithmetic are rather difficult to classify since the preponderance of Latin terms corresponds not to Roman interest in arithmetic but rather to the use of Latin as the universal language of educated people in Europe until the eighteenth century or

later. Even the early American universities continued for a time the use of textbooks in Latin.

Both Greek and Latin forms are used to construct technical terms which are entirely modern. The word "telephone" is based on Greek stems; similarly the word "trigonometry" is Greek in form, but it is a constructed word appearing first in a Latin work of 1590. In arithmetic there are several such constructed terms. Thus "fraction" was used by Leonard of Pisa in 1202, and in two twelfth-century Latin translations of Arabic arithmetical works, being a translation into Latin of the Arabic term. Neither Caesar nor Cicero knew the term, and the fundamental concept back of the word is Arabic and not Latin.

The words which represent ideas not technical but necessary in everyday affairs are largely from a parent language preceding Greek and Latin. Several Old English (Anglo-Saxon, old Saxon) terms are included in this group.

Influence of other races. The Hindu and Arabic terms are few in number but significant in meaning and in import. Spanish or Dutch influence is evident only rarely, if at all, in arithmetical terminology, and German rarely. French terminology connects most closely with the English, while the Italian offers quite a few terms in business arithmetic.

The Origin of Number Names

Number words. The word "number" comes to us through the Latin *numerare*, meaning "to count," and more directly through the French *nombre*. The stem connects with a Greek word having the same significance,

and probably both have a common origin. The separate words designating numbers from one to one thousand connect directly, so far as we know, with the earliest forms which were used to designate numbers in the

[14]

Tare and Tret.

Tare is an allowance made to the buyer for the weight of the hogſhead, barrel, box, or whatever elſe contains the goods bought, and is calculated at ſo much per hogſhead, barrel, &c. or at ſo much per cent. or at ſo much in the groſs weight.

Tret is an allowance made to the buyer of 4 pounds in 104. for waſte and duſt in ſome ſorts of goods.

112 pounds weight is call'd a groſs hundred, and 100 pounds a neat hundred; ſome ſorts of goods are ſold by one weight and ſome by the other. When an article is ſold by groſs hundreds, the price is generally ſpecified at ſo much per hundred, and the tare per cent. is upon 112 pounds. When an article is ſold by neat hundreds, the price is generally ſpecified at ſo much per pound, and the tare per cent. is upon 100 pounds.

The whole weight of an article, and the hogſhead or whatever contains it, being weighed together, is called the groſs weight, whether the article be ſold by groſs hundreds or neat hundreds.

The weight of the article itſelf, after all allowances are deducted, is called the neat weight, whether the article be ſold by groſs hundreds or neat.

Caſe 1ſt. When the tare is at ſo much per hogſhead, barrel, &c. multiply the number of hogſheads or barrels by the tare, and the product will be neat hundreds; reduce this product to groſs hundreds if the article is ſpecify'd in groſs hundreds, and ſubtract it from the groſs weight; the remainder is the neat weight.

Caſe 2d. When the tare is at ſo much per cent. and is the aliquot part or parts of an hundred weight, divide the whole groſs by the ſaid part or parts which the tare is of an hundred weight; the quotient thence ariſing

gives

[24]

Rebate or Diſcount.

Rebate or diſcount is when a ſum of money due at any time to come, is ſatisfy'd by paying ſo much preſent money, as being put out to intereſt, would amount to the given ſum in the ſame ſpace of time.

Find the amount of £100 for the time and rate per cent. given, which intereſt add to £100; then by a ſtating in the rule of three ſay, as that ſum is to £.100 ſo is the debt or ſum propoſed to the preſent worth required. The difference between the preſent worth and the given ſum is the rebate.

Equation of Payments.

When ſeveral ſums of money are to be paid at different times, and it is required at what time the whole ſhall be paid together, without loſs to debtor or creditor; this is called equation of payments, or equating the time of payment. Multiply each payment by its time, add the products together, and divide this ſum by the whole debt, the quotient is the equated time.

Fellowſhip.

By Fellowſhip the accompts of ſeveral partners, trading in a company are ſo adjuſted or made up, that every partner may have his juſt part of the gain, or ſuſtain his juſt part of the loſs; according to the proportion or ſhare of money he hath in the joint ſtock. There are two kinds of fellowſhip, viz. ſingle and double. Single fellowſhip is when the ſtocks of all the partners continue an equal term of time, and is uſually call'd fellowſhip without time. Double fellowſhip is when the ſtocks continue an unequal term of time, and

is

BENJAMIN DEARBORN, "THE PUPIL'S GUIDE," BOSTON, 1782
(*First edition same year in Portsmouth*)

Tare and trett continued to appear for many years in American textbooks. Our word "tariff" and the word "tare" are Arabic in origin.

period of the formation of the Indo-European languages. These words indicate a common prehistoric source of all European languages and of many Asiatic, including particularly the Sanskrit group. To this group belong the words *one*, *two*, *three*, ... *eleven*, *twelve*, *thirteen*, the *-teens*, the *hundreds*, and the *thousands*.

Particular attention is directed to the fact that eleven and twelve, from *un-lif* and *zwo-lif*, are out of harmony with the other *-teens*. The score (20) is an early form appearing in Old English; in French eighty and ninety are given in terms of scores, *e.g.*, *quatre vingt dix*, or four scores and ten, for ninety. "Dozen" is somewhat later, a Latin derivative. The English numeral words, notably "hundred" and "thousand," resemble the German and Scandinavian forms rather than the Latin forms, *centum* and *milia*, used in the Romance languages. However, many common words, like "century," "cent," "mile," "millenium," employ the Latin form.

"Million," "billion," and "trillion" are terms which have come into popular use largely since the Great War. The words are comparatively recent, appearing in the fifteenth century. Million appears first in print in Borghi's arithmetic of 1494, an Italian form implying "greater thousand." Concerning trillion, quadrillion, and the like, English and American use differs from continental. In America we use "billion" for one thousand millions and "trillion" for one thousand billions. However, continental writers largely follow Chuquet, the Frenchman, who used these terms and "million" in a manuscript of 1484, interpreting billion as the square of 1,000,000, and trillion as the cube. The French occasionally use *milliarde* with our meaning of billion. Isaac Greenwood in 1729 and Nicholas Pike in 1788 give "billions" as "millions of millions," and also employed "trillions," "quadrillions," etc., in the continental sense.

"Zero" and "cipher" are directly Arabic in their origin, and appear in the early arithmetics explaining the

new numerals. "Decimal point" is much more recent, as indicated above, having been preceded in England and America by the word *separatrix*, which is used by writers like Recorde, Greenwood, and Pike.

Operative Terms and Symbols

Latin influence. Instruction in European church schools and in the universities was carried on in Latin until the seventeenth century. In consequence the technical terminology of arithmetic, algebra, geometry, and trigonometry is derived from the Latin used in these schools, as opposed to classical Latin. To distinguish between the technical use of any given term and the ordinary non-technical use is not easy, nor would authorities always agree.

The words "add," "subtract," "divide," and "multiply," and others too numerous to list, correspond to classical Latin forms. However, as systematic exposition of the fundamental operations in arithmetic is not given in Latin before the tenth- and eleventh-century treatises on the abacus, the strict technical use of these terms is largely not found until this time. In the theoretical arithmetical treatise of Boethius which does not touch the fundamental operations, in the architecture of Vitruvius (*c.* 50 B.C.), in the work on aqueducts by Frontinus (*c.* 90 A.D.), and in other Latin practical treatises of the first centuries of the Christian Era, the terminology of our arithmetic has its beginning.

Words derived from Latin terms used by Boethius and earlier writers with implication of the modern meaning are illustrated by the list on the following page.

addition (Boethius used several other terms as well for this operation)

subtraction	line	quantity
multiplication	quadrilateral	product
prime	triangle	minus
unit	perpendicular	sum
digit	plane	minutes
altitude	angle	seconds
quotient	proportion	equal

Latin forms introduced with the new arithmetic and algebra in the twelfth century and later are illustrated by the following list:

fraction (Leonard of Pisa, 1202 A.D.; John of Spain, Latin translation of Al-Khowarizmi's arithmetic, twelfth century)

numerator	minuend	equation
denominator (1484, French)	subtrahend	dividend
	division	quotient
vulgar fraction	divisor	quadratic
common fraction	multiplier	multiplicand
remainder	decimal	abstract
integer	plus	concrete
multiple	minutes	seconds
factor	improper (fraction)	notation
exponent		

surd (Gerard of Cremona, 12th century)
radius (Fink, 1583, in German)
rectangle (Mersenne, *c.* 1620, in a French work)
coefficient (Vieta, 1591)

The arithmetical terms above appear largely in the two fifteenth-century arithmetics in English, and definitely established in Recorde's *Grounde of Artes* and Humphrey Baker's *Wellspring of Sciences.* The algebraical terms are found largely in English in Recorde's *Whetstone of Witte* and the geometric to some extent

in his *Pathwaie to Knowledge*, but much more commonly in the first English translation (in print) of Euclid, Billingsley's *Euclid*, published in 1570.

Greek terms. On the other hand, words of Greek origin which connect directly with Euclid, Archimedes, Plato, and other Greek mathematicians include the following representatives:

problem	geometry	cube
parallel	base	ellipse
stereometry	center	parabola
diameter	cone	hyperbola
axis	theorem	polygon
analysis	isosceles	orthogonal
synthesis	periphery	rhombus
perimeter		

Arabic terms. The contributions of the Arabs to the terminology of elementary mathematics touch also many contributions of the Hindus. The more fundamental words of this group are included in the following list of words of Arabic and Hindu origin:

algebra Meaning "the restoration," referring to the transference of a negative term from one side of an equation by the addition of the corresponding positive term, *e.g.*, $10x-x^2=21$, by the operation of "algebra" becomes $10x=21+x^2$.

algorism Transliteration into Latin of Al-Khowarizmi's name; long used with the meaning "arithmetic."

cipher, zero Both from Arabic *sifr*, meaning "vacant," which is derived from the Sanskrit *sunya*, meaning "vacant."

degree Of an angle; from the Arabic term.

sine From the Latin *sinus*, which was a translation of the Arabic word *el-gaib*, meaning "curve," which in turn was a transliteration of a part of the term used in India to mean *sine*.

tariff, tare	From the Arabic business arithmetic.
azimuth	From Arabic astronomical works.
radix, root	Translation of an Arabic word *gidr*, which in turn is the translation of the Sanskrit word *mula*, meaning "root" (vegetable) and "square root" of a number. Used also by the Arabs in the sense of root of an equation.
radical	Appears in Clavius, *Algebra*, 1608; probably Arabic influence.
fraction	Used in Latin translations of Arabic works; translation of Arabic word meaning "broken number," which latter term was long used in English texts as explanatory to "fractions."

Chaucerian usage. The first writer to introduce into English a large number of technical terms was Chaucer in the second half of the fourteenth century. In his treatise on the *Astrolabe* and in the popular *Canterbury Tales* Chaucer uses many technical terms, largely following French forms.

Among these words introduced or made popular by Chaucer are the following:

adden	degree	infinit
adding	diameter	latitude
altitude	diminucioun (1303)	longitude
angle	divisioun	millioun
calculinge	doseyn (1300)	multiplicacioun
centre	double	perpendiculeer
circle	egal (equal)	proporcioun
circumscryve	emispherie	serie
compasse	encrees (increase)	divyde
consentrik		

Of the above words many are also used in English by Gower and Wiclif in the same century; the Chaucerian

11

spellings are retained above. Chaucer used also the following terms of measurement: mesure (1200 in English), galoun (1300), ounce, quart, busshel (1300), myle, barel, minutes, and secoundes.

Norman French influence. Many words of Latin and Greek origin were introduced into the English language through the mediation of French from the time of the Norman conquest well into the fifteenth century. In England during this period French was the common language of the educated classes. In the universities Latin was used for a longer period of time, but the terminology of arithmetic in English was largely fixed by the end of the fifteenth century. In algebra and geometry, texts in English did not appear until in the sixteenth century, and the common terms in these subjects date from that time. The printed works on arithmetic by Recorde, Humphrey Baker, and Leonard Digges probably reflect a terminology already well established in mathematical circles, particularly in the separate "ciphering" schools. Attention has been called (*see* page 142) to the fact that many of the business terms of arithmetic are Italian in origin.

The use of the dictionary. The terms of mathematics may well be used by a teacher in instructing high-school children in the extended use of the dictionary. The history of mathematics in its major outlines is reflected in the terminology. Under the wise direction of a good teacher a child may himself discover much of this history through the aid of the dictionary. Real appreciation of the historical development of the mathematical sciences can be obtained by this simple method.

Bibliography for Supplementary Reading

James A. H. Murray (editor), *A New English Dictionary*, on historical principles. Oxford, 1884 to date.

This remarkable historical dictionary is the achievement of a large group of scholars. Under each word are found illustrations of the use of the word from its first known appearance down to recent times. The teacher of mathematics, no less than the teacher of English and history, will obtain a new appreciation of the development of the English scientific vocabulary by examining this work.

The preface and the chapter, "A Brief History of the English Language," in Webster's *New International Dictionary* (1917), should be read by the teacher using this excellent dictionary.

Greenough and Kittredge, *Words and their Ways in English Speech*. New York, Macmillan Co., 1901.

W. W. Skeat, *An Etymological Dictionary of the English Language*. Numerous editions. The introductory material is particularly worth reading in connection with the etymology of mathematical terms.

CHAPTER VIII

DENOMINATE NUMBERS

General Considerations

From concrete to abstract arithmetic. The use of numbers with concrete objects in measuring and weighing and counting doubtless constitutes the earliest step in the development of mathematical ideas. So far as integers are concerned, the step from the abstract to the concrete precedes any historical record. However, in the development of fractions among the Babylonians and among the Romans the abstract fractions were directly derived from concrete units which continued in use. The fractions of the Babylonians survive in our minutes and seconds; those of the Romans survive in our inches and ounces and in apothecary weights and measures.

Time

Beginning of the day. The most natural unit for measurement of time is the length of the day, from sun to sun. This day the Babylonians took from sunrise to sunrise, while the Hebrews and the Greeks reckoned the day from sunset to sunset. Modern business and law follow Roman procedure in beginning the day at midnight, while astronomers find it more convenient to follow the Arabs in taking high noon as the starting point.

Subdivision of the day. The Babylonians divided the day into twelve hours as recorded by equal divisions on a sundial; the night was conceived as divided into twelve

corresponding parts. This twenty-four-hour day is Babylonian in origin; twenty-four divisions equal in length were first established by the Greeks.

To the Babylonians is due also the division of the hour into sixty minutes and of the minute into sixty seconds. In circular measurement the degree was correspondingly subdivided, and the Latin translations used the terms *partes minutae* (or *minutiae*) *primae* and *partes minutae secundae*, whence our minutes and seconds. The use of terms corresponding to these does not appear however before the ninth century among the Arabs, and doubtless appeared in Europe first in Latin translations of Arabic works.

The week and the month. To the Babylonians is traced also our week of seven days, corresponding respectively to the seven planets: Sun, Moon, Mars, Mercury, Jupiter, Venus, and Saturn. The origin of our words, "Sunday," "Monday," and "Saturday," is evident.

The month, as the word indicates, is connected with the passage of the moon about the earth. The word is an old Anglo-Saxon one.

MENSURATION

Origin of units of measure. The primitive measures of length are those derived from the human body; the foot, the digit, the palm, the span, the ell (elbow) or cubit, the *pouce* or thumb, and the pace appear as units of measure in the earliest records of civilized peoples, and similar natural units are in use among primitive peoples today. Scientific systems of measures and of weights and moneys, as we shall see, began with these primitive

forms, modifying them in accordance with developments requiring greater precision.

Babylonian linear units. The earliest scientific units of linear measure known to us are the Babylonian units, which correspond in idea most remarkably to the metric system of the French. The Babylonians, as we have shown in a preceding chapter, based their number system upon sixty. This same base appears constantly in their systems of weights and measures.

Babylonian Table

3 lines = 1 sossus
10 sossus = 1 palm
3 palms = 1 small ell (or cubit)
5 palms = 1 large ell
6 large ells = 30 palms = 1 large seed
60 palms = 1 gar
60 gar = 1 ush (or stadion)
30 ush = 1 kask or parasang

The palm appears to have measured approximately four inches; the twelfth of a palm was also used as a measure, and also the third of a palm, or a digit. The measures of area were based upon the squares of the above units. Most striking is the fact that the cubic palm, or *ka*, was taken as a measure of capacity, closely approximating the liter and quart, while the weight of one *ka* of water was taken as the unit of weight, one *mina*.

French development parallels the Babylonian. This ancient Babylonian procedure corresponds precisely to the French procedure, 3500 years later, in establishing a connection between linear and cubic measure, and a unit of weight. The parallel is one of the most striking to be found in the development of scientific ideas. Later

A Comparifon of the American foot with the feet of other Countries.

The American foot being divided into 1000 parts, or into 12 inches, the feet of feveral other Countries will be as follow.

	Parts.		Inch.	lin.	points.
America —	1000	— — —	12	0	0 *dec.*
London —	1000	— — —	12	0	0
Antwerp —	946	— — —	11	4	1,32
Bologna —	1204	— — —	14	5	2,25
Bremen —	964	— — —	11	6	4,89
Cologne —	954	— — —	11	5	2,25
Copenhagen —	965	— — —	11	6	5,76
Amfterdam —	942	— — —	11	3	3,88
Dantzick —	944	— — —	11	3	5,61
Dort —	1184	— — —	14	2	2,97
Frankfort on the main	948	— — —	11	4	3,07
The Greek —	1007	— — —	12	1	0,04
Lorrain —	958	— — —	11	5	5,71
Mantua —	1569	— — —	18	9	5,61
Mecklin —	919	— — —	11	0	2,01
Middleburg —	991	— — —	11	10	4,22
France —	938	— — —	11	3	0,43
Prague —	1026	— — —	12	3	4,46
Rhyneland or Leyden	1033	— — —	12	4	4,51
Riga —	1831	— — —	21	11	3,98
Roman —	967	— — —	11	7	1,48
Old Roman —	970	— — —	11	8	0
Scotch —	1005	— — —	12	0	4,32
Strafburgh —	920	— — —	11	0	2,88
Toledo —	899	— — —	10	9	2.73
Turin —	1062	— —	12	8	5 66
Venice —	1162	— —	13	11	1,96

A Table reprefenting the conformity of the weights of the principal trading Cities of Europe with thofe of America.

lb			*of America.*	
100 *of*	*England, Scotland and Ireland* — —	Equal	100*lb.*	*oz.*
100 *of*	*Amfterdam, Paris, Lourdeaux, &c.*	——	109	8
100 *of*	*Antwerp, or Brabant* ——	——	103	12
100 *of*	*Rouen, the Vifcounty* ——	——	113	14
100 *of*	*Lyons, the City* —— ——	——	94	3
100 *of*	*Rochelle* —— ——	——	110	9
100 *of*	*Touloufe, and upper Languedoc* —	——	92	6
100 *of*	*Marfeilles and Provence* ——	——	88	11
100 *of*	*Geneva* —— ——	——	123	
100 *of*	*Hamburg* —— ——	——	107	5
100 *of*	*Francfort* —— ——	——	111	11
100 *of*	*Leipfic* —— ——	——	104	5

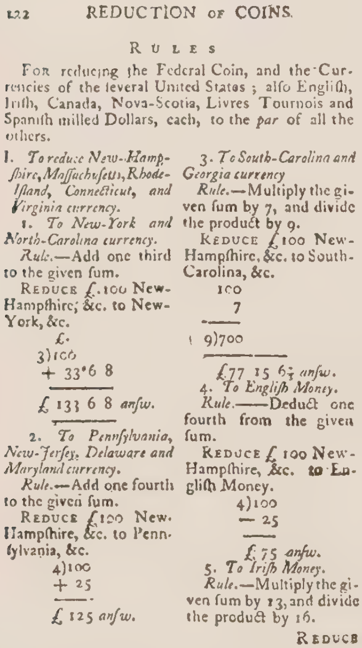
A

PIKE'S "ARITHMETICK," 1788

Early difficulties with weights and measures.

among the Egyptians and among the Greeks the attempt was made to establish similar connections for unity of weights and measures.

The *mina* in Babylon was subdivided into sixty *shekels*, and each of these into 360 *she*, or grains of corn. Originally the *shekel* was a weight, but in Babylon, as later in Greece and Rome, the term soon became employed as a unit of money.


122 REDUCTION OF COINS.

RULES

FOR reducing the Federal Coin, and the Currencies of the ſeveral United States; alſo Engliſh, Iriſh, Canada, Nova-Scotia, Livres Tournois and Spaniſh milled Dollars, each, to the *par* of all the others.

1. *To reduce New-Hampſhire, Maſſachuſetts, Rhode-Iſland, Connecticut, and Virginia currency.*

1. *To New-York and North-Carolina currency.*

Rule.—Add one third to the given ſum.

REDUCE £.100 New-Hampſhire, &c. to New-York, &c.

£.
3)100
+ 33 6 8
£ 133 6 8 *anſw.*

2. *To Pennſylvania, New-Jerſey, Delaware and Maryland currency.*

Rule.—Add one fourth to the given ſum.

REDUCE £100 New-Hampſhire, &c. to Pennſylvania, &c.

4)100
+ 25
£ 125 *anſw.*

3. *To South-Carolina and Georgia currency*

Rule.—Multiply the given ſum by 7, and divide the product by 9.

REDUCE £100 New-Hampſhire, &c. to South-Carolina, &c.

100
7
9)700
£77 15 6$\frac{2}{3}$ *anſw.*

4. *To Engliſh Money.*

Rule.——Deduct one fourth from the given ſum.

REDUCE £ 100 New-Hampſhire, &c. to Engliſh Money.

4)100
— 25
£ 75 *anſw.*

5. *To Iriſh Money.*

Rule.—Multiply the given ſum by 13, and divide the product by 16.

REDUCE

CURRENCY TROUBLES FROM PIKE'S "ABRIDGEMENT" OF 1793

In 1788 Pike printed £.100 instead of £100 as here shown. It is evident that the early colonists had currency difficulties somewhat analogous to those troubling Europe today.

George Washington wrote in praise of Pike's work of 1788 as follows: "The handsome manner in which that work is printed and the elegant manner in which it is bound, are pleasing proof of the progress which the Arts are making in this Country. . . . The investigation of mathematical truths accustoms the mind to method and correctness in reasoning and is an employment peculiarly worthy of rational beings."

Quite probably the measures and weights of Egypt were based in ancient times upon those of Babylon. Certain it is that these orientals directly influenced the Greek, and thus the Roman, systems of weights and measures.

The mile. The Roman foot, *pes*, with plural, *pedes*, is now determined as having been slightly less than the English foot; 5 feet gave the Roman *passus* or pace, and *milia pasuum*, or 1000 paces, gave the Roman mile, about 95 yards shorter than our mile.

LABORED EXPLANATION OF FEDERAL MONEY IN PIKE'S "ABRIDGEMENT" OF 1793

There are only slight variations in this discussion from that given in the "Complete System of Arithmetick" of 1788, which quotes the Act of Congress "the 8th of August 1786" establishing the federal money.

The Coinage Act of April 2, 1792, formally established our present system in the essential details concerning the units. The silver dollar was first coined under the Act in 1794.

FEDERAL MONEY. 103

As the Money of Account proceeds in a decuple, or tenfold, proportion, ſo, any number of Dollars, Dimes, Cents and Mills, is, ſimply the expreſſion of Dollars and Decimal parts of a Dollar :—Thus 9 Dollars and 8 Dimes are expreſſed 9,8 = $9\frac{8}{10}$ *doll.*—12 Dollars, 4 Dimes and 7 Cents, thus, 12,47 = $12\frac{47}{100}$ *doll.* 20 Dollars, 3 Dimes, 4 Cents and 5 Mills, thus 20,345 = $20\frac{345}{1000}$ *doll*—100 Dollars and 9 Mills, thus, 100,009 = $100\frac{9}{1000}$ *doll.* and 50 Dollars, 5 Cents, thus 50,05 = $50\frac{5}{100}$ *doll.* wherefore, it is, in all reſpects, added, ſubtracted, multiplied and divided, the ſame as Decimals ; and, of all Coins, it is the moſt ſimple.

			marked.
Note. 10 Mills	*make one*	Cent.	*m. c.*
10 Cents		Dime.	*d.*
10 Dimes		Dollar.	*D.*
10 Dollars		Eagle.	*E.*

ADDITION *of the* FEDERAL MONEY.

ADD 25½ Eagles ; 7 Dollars, 8 Dimes, 3 Cents, 4 Mills : 125 Dollars, 8 Cents ; 5 Eagles, 9 Mills ; 18 Dollars, 7 Cents and 4 Mills together.*

* IT may be obſerved that the ſum exhibits the particular number of each different piece of money contained in it. viz. 455997 Mills = $45599\frac{7}{10}$ Cents = $4559\frac{97}{100}$ Dimes = $455\frac{997}{1000}$ Dollars = $45\frac{5997}{10000}$ Eagles = 4 5 5, 9 9 7. (*E D. d. c. m.*)

ALSO, the names of the Coins, leſs than a dollar, are ſignificant of their values.

FOR the *Mill*, which ſtands in the 3d place, at the right hand of the comma, or place of thouſandths, is contracted from *Mille*, the Latin for *Thouſand* :—*Cent*, which occupies the ſecond place, or place of Hundredths, is an abbreviation of *Centum*, the Latin for *Hundred* :—and *Dime*, which is in the firſt place, or place of tenths, is derived from *Diſme*, the French for *Tenths*.

Natural measures of weight. The primitive system of weights takes as its ultimate unit the barleycorn or the grain of wheat, or the seed of the carob (a plant), whence we obtain the "carat" used in weighing gold and diamonds.

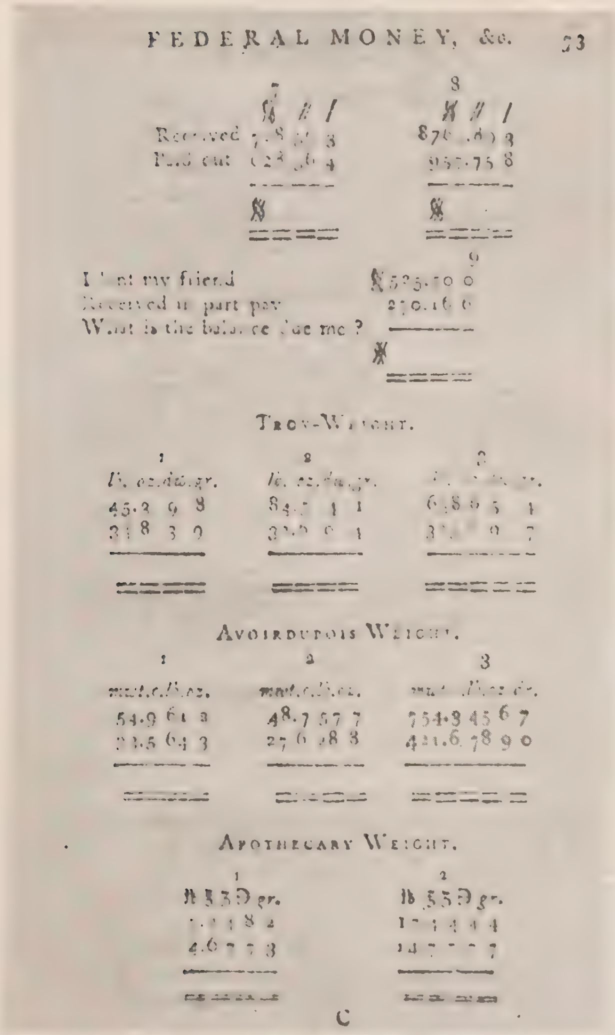
Dry and liquid measure. The Romans employed different systems for dry and liquid measure which, in

altered form, continue with us to the present day. The cubic foot, termed an *amphora*, was the fundamental unit of liquid measure, and one-third of it, or *modius*, the unit of dry measure. The *congius*, of which eight make an *amphora*, is about three-quarters of a gallon; the *modius* is very nearly one peck.

The systems of avoirdupois and Troy weights are directly French in origin, based on Roman and modified by early Saxon forms. In particular Troy weight probably referred originally to weights used by jewelers in the French city of Troyes.

Origin of inches and ounces. The Roman foot or *pes* was divided into twelve *unciae*, whence our inches. The Roman unit of weight, a bar one foot in length, was divided into sixteen smaller units also called *unciae* (*uncia*), whence our ounces.

Varying standards. In early England up to 1400 A.D. there were, as on the continent of Europe, varying standards for the foot and for the pound. However, the most common foot was probably that which measures 13.22 of our inches. By the statute of the Assize of Bread and Ale in 1266 the following table was established: "An English penny called a sterling, round and without any clipping, shall weigh thirty-two wheat corns in the midst of the ear; and twenty pence do make an ounce, and twelve ounces do make a pound; and eight pounds do make a gallon of wine, and eight gallons of wine do make a bushel."

The American colonies continued largely the use of the English units. However, after the Revolutionary War no less personages than Washington and Jefferson


FEDERAL MONEY, &c. 73

Received
Paid out

I lent my friend
Received in part pay
What is the balance due me?

TROY-WEIGHT.

AVOIRDUPOIS WEIGHT.

APOTHECARY WEIGHT.

C

CHAUNCEY LEE, "THE AMERICAN ACCOMPTANT," LANSINGBURGH, 1797
First appearance in print of the dollar sign.

attempted to introduce decimal systems of weights and measures as well as of moneys.

In linear and square measure and in weights our units correspond to the English. However, the English imperial gallon and quart are fully twenty percent larger than the American gallon and quart; the imperial gallon contains 277.274 cubic inches as opposed to 231 in an American gallon. Similarly, the English imperial bushel contains 2718.192 cubic inches as opposed to 2150.42 in the American legal bushel.

The Metric System of Weights and Measures

Metric system. The desirability is evident of some unit of length established with reference to some fairly unchangeable natural distance upon the earth's surface or in nature. More than one hundred years before the French Revolution Gabriel Mouton of Lyons, France, proposed that the arc of one minute of a great circle of the earth should be taken as the *mile*, of which the thousandth part should be the unit of length. Shortly afterwards the famous astronomers Picard (in 1671) and Huygens (in 1673) proposed the length of a pendulum beating seconds as the universal yard. In the eighteenth century numerous proposals along these general lines were made, particularly in France.

Long before Mouton's proposals of 1670, the Flemish mathematician and scientist, Simon Stevin of Bruges, published, first in Flemish and in the same year, 1585, in French, a treatise in which the first explanation of decimal fractions is given. In this same treatise Stevin

proposes that not only weights and moneys but also linear, square, and cubic measure and even degrees and minutes should be reduced to a decimal system. This proposal, together with the explanation of the decimal fractions, establishes for Simon Stevin a proud place in the history of the development of scientific systems of measurement.

Talleyrand in 1790 brought the matter of uniform systems of weights and measures to the attention of the French National Assembly. In October of 1790 a committee on which mathematicians were represented by Lagrange and Laplace reported favorably upon the desirability of a decimal system of weights, measures, and moneys. In 1791 it was agreed that the unit of length should be one ten-millionth part of the quadrant of a meridian. In 1795 the *meter*, as unit of length, the *are* (100 square meters) as unit of area, the *liter* as unit of volume, and the *gramme* or *gram* as unit of weight were formally adopted, together with the *franc* as monetary unit. In the course of the scientific determination of the unit of length and of weight the great scholars of France were assisted by Danish, Swiss, Spanish, and other European scholars. Incidental to the final determination of the gram the physicists Lefèvre-Gineau and Fabbroni discovered that the maximum density of water is reached at 4° Centigrade.

Uniform measures desired. In America Washington recognized in two annual messages to Congress the great desirability of uniformity in currency, weights, and measures. Thomas Jefferson, in a "Report on Weights, Measures and Coinage" submitted in 1790, urged the

Schoolmaster's Assistant. 125

3. Required the Interest of 94l. 7s. 6d. for one year, five months and a half, at 6 per cent per annum.

Ans. 8l. 5s. 1d. 3,5qrs.

4. What is the interest of 12l. 18s. for one third of a month, at 6 per cent? *Ans. 5,06d.*

---

2. *For Federal Money.*

RULE.

Divide the principal by 2, placing the separatrix as usual, and the quotient will be the interest for one month, in cents, and decimals of a cent; that is, the figures at the left of the separatrix will be cents, and those on the right decimals of a cent.

2. Multiply the interest of one month by the given number of months, or months and decimal parts thereof, or for the days take the even parts of a month, &c.

EXAMPLES.

1. What is the interest of 341 dols. 52 cts. for $7\frac{1}{2}$ months?

2)341,52
170,76 Int. for 1 month.
$7\frac{1}{2}$
1195,32 ditto for 7 months.
85,38 ditto for $\frac{1}{2}$ month.
1280,70 *Ans.* 1280,7 *cts.* = 12*dols.* 80*cts.* 7*m.*

Or thus, 170,76 Int. for 1 mo.
× 7,5 months.
85380
119532
D.dcm.
1280,700 cts.=12,807

2. Required the interest of 10 dols. 44 cts. for 3 years, 5 months and 10 days.

2)10,44
10 days = $\frac{1}{3}$) 5,22 Interest for 1 month.
41 months
5,22
208,8
214,02 ditto for 41 months.
1,74 ditto for 10 days.
215,76 *cts. Ans.*=2*dols.* 15*cts.* 7*m.* +

L 2

DABOLL'S "SCHOOLMASTER'S ASSISTANT," NEW LONDON, 1802

First appearance in print of the six-percent rule.

change to a decimal system. James Madison in his "Annual Message of 1816" and John Quincy Adams in a "Report on Weights and Measures" presented in 1821 urged the extension of the decimal system of coinage to similar decimal weights and measures. Not until 1866, however, was the metric system made legal; at that time the yard was replaced by the meter, as official standard, in that the yard was legally defined as bearing the ratio 3600 to 3937 to the meter. Every year sees further progress towards the general adoption of the metric system, so that one may reasonably hope that before another century appears the prophecy of John Quincy Adams in 1821 will be fulfilled that "the meter will surround the globe in use as well as in multiplied extension; and one language of weights and measures will be spoken from the equator to the poles."

Money

Federal money. The dollar currency was established by an Act of Congress of 1786. However, before 1775 the dollar was frequently employed, having reference to the Spanish dollar, which was widely used even as late as 1850. So far as the textbooks of arithmetic were concerned, the bulk of the problems involving money continued to use pounds well to the end of the eighteenth century. American dollars were first coined in 1794.

Our symbol for the dollar has been shown very clearly by Professor Cajori to be a transformation of the symbol for pesos employed in dealing with Mexico and South America and long current in colonial America as Spanish dollars.

In early colonial America English pounds and guineas were more or less standard. However, French guineas, Dutch or German ducats, Spanish pistoles and pesos, and a dozen other types of coins were common. The early American almanacs frequently include tables of the value and weight of coins that were common in the colonies.

Bibliography for Supplementary Reading

Florian Cajori, "The Evolution of the Dollar Mark," *Popular Science Monthly*, December, 1912 (Vol. 81, pp. 521–530).

W. H. Hallock, *Outline of the Evolution of Weights and Measures and the Metric System*. New York, Macmillan Co., 1906.

See also the references at the end of Chapter VI.

See encyclopedias under Calendar, Weights and Measures, Metric System, Dollar, and Money.

CHAPTER IX

THE TEACHER AND THE TEACHING OF ARITHMETIC

Egyptian priests. The Egyptian priests were devoted students of the mathematical sciences. Undoubtedly to them was confided the instruction of the Egyptian children in arithmetic and geometry, as Greek writers indicate. The methods of the Egyptians are given high praise by Plato, who states (*see* page 137) that the Egyptians taught their children arithmetic by means of games, with apples and nuts and bowls. This testimony of Plato bears witness to the ability of the Egyptian teachers, and to this pedagogical gift the practical problems of the Ahmes papyrus and other ancient documents from the land of the Nile bear witness.

Classical pedagogues. The pedagogue in Greece was the slave who accompanied the child to and from school. Both in Greece and in Rome elementary instruction including numbers was frequently given by such a slave. However, the teacher of arithmetic and more particularly of geometry enjoyed a higher status. In two points the classical tradition has continued practically to the present day: the teachers of Greece and Rome were poorly paid and their instruction was supplemented by a liberal use of the rod (or ferrule). Illustrations from the classical period represent the unfortunate subjects of instruction receiving corporal punishment at the hands of the teacher. Theoretical arithmetic was studied in Greece by adults as preparatory to philosophy. Our knowledge of the

instruction of children in computation is fragmentary, depending upon chance references and not upon any systematic Greek account of the subject.

Medieval instruction. In the church schools of the Middle Ages arithmetic was included largely for the computation of Easter; the technical treatise on this latter topic was called a *computus*. So far as arithmetic itself is concerned Boethius was the author whose text was widely used. In general this meager instruction in the mathematical sciences was given as an extra on feast days and on holidays. The most famous teacher of the early Anglo-Saxons was the Venerable Bede (*c.* 673–735), to whom is credited a *computus* dealing with the determination of Easter, and a treatise on reckoning with the fingers. In his course of study for priests arithmetic is given a proper place.

Public education had a faint beginning in the church schools which are connected with the names of the educator Alcuin, born in 735, and the Emperor Charles the Great whom Alcuin greatly influenced. The Capitulary of 789 A.D. designates arithmetic as one of the subjects to be taught to children in the schools attached to religious foundations, and in such a way some instruction in elementary arithmetic was given to children in many parts of Europe from the ninth to the fifteenth centuries. In spite of this beginning, practical arithmetic from the twelfth to the sixteenth century was commonly taught by laymen outside the schools, in a way similar to modern instruction in music and dancing. In a few universities lectures on the new Hindu-Arabic arithmetic were given, commonly following the algorisms of Sacrobosco or that

of Alexandre de Ville Dieu. The lecturer would read a few lines of the text, following this with a long disquisition upon the passage read.

Reckoning schools. In Germany and Holland the *Rechenmeister* was appointed by the city to act as town clerk and was given a practical monopoly of the business of instruction in arithmetic. Frequently it was contracted that the *Rechenmeister* and city clerk should give instruction in arithmetic in the Latin schools where Latin and Greek constituted the principal subjects of instruction. The salary for instruction in reading, writing, and reckoning was frequently paid "in kind," and the teacher eked out an existence by supplementary tasks as sexton, bell ringer, or by attendance at the house of the wealthy or noble. These unfortunate characteristics of the profession marked also, as we shall note, the status of the teacher in early American schools.

Instruction in England. In English grammar schools arithmetic was rarely taught, but appears in private schools for a separate fee. Humphrey Baker presents in his arithmetic of 1562 a typical advertisement: "Such as are desirous, eyther themselves to learn or to have theyr children or servants instructed in any of these Arts and Faculties heere under named: It may please them to repayre unto the house of Humphrey Baker, dwelling on the North side of the Royall Exchange, next adjoyning to the signe of the shippe. Where they shall fynde the Professors of the said Artes, etc., Readie to doe their diligent endeavours for a reasonable consideration. Also if any be minded to have their children boorded at the said house, for the speedier expedition of their learning,

ARTS *AND* SCIENCES

MATHEMATICALL

TAught in Fetter-lane neare the golden Lyon, or privatly abroad at convenient houres, by *Robert Hartwell* Teacher of the *Mathematicks*.

Viz. { Arithmetick, Geometry, } Astronomy, { Cosmography, Geography, Navigation, Architecture, Fortification, Horologiography, &c. }

Measuring of Land.

The doctrine of Triangles { plaine and sphericall. } the use of the Tables of { Sines, Tangents, Secants, & Logarithmes. }

Accompts for *Merchants* by order of Debitor and Creditor.

Fide *Sed* *Vide.*

Vivat Rex.

"MR. BLUNDEVIL HIS EXERCISES CONTAYNING EIGHT TREATISES"
From the seventh edition, revised by R. Hartwell, London, 1636.

they shall be well and reasonably used, to theyr contentation."

Numerous such advertisements appear in English arithmetics of the seventeenth century and even of the eighteenth century. Such a teacher was James Hodder, whose arithmetic of 1661, London, was reprinted in Boston in 1719. Hodder kept "a school in Lothbury next door to the 'Sunne,' where such as are desirous to learn . . . Writing, as also Arithmeticke in whole numbers and Fractions, with Merchants' accompts and Shorthand, may be carefully attended and faithfully introduced. . . ." Thomas Dilworth in 1743 lists for advertising purposes fifty English teachers of arithmetic who recommend his book, long popular in England and equally so in the American colonies from 1750 to the beginning of the nineteenth century.

Early American instruction. In colonial America instruction including arithmetic received the early attention of the English and the Dutch settlers, while the Spanish were more occupied with the conversion of the natives to Christianity and of gold and silver to Spain. Towards the end of the seventeenth century "to cipher and to cast accounts" appears in the schools of New England, and a little later in the schools of New York and Pennsylvania.

As early as 1645 the master of the "free school" in Boston was allowed fifty pounds and house, while "an usher, who should also teach to read and write and cipher," was allowed thirty pounds, "the charge to be by yearly contribution either by voluntary allowance, or by rate of such as refused, . . . " At Dedham in

Massachusetts Jacob Farner was appointed in 1653 to teach writing and reading and "the art and knowledge of Arithmetick and the rules and practice thereof"; his salary was fixed at twenty pounds per year. Three years later Recorde's arithmetic is known to have been used in the schools of Dedham by Michael Metcalfe.

Private schools played a more important rôle in the eighteenth century than now, probably because of the neglect of education on the part of many towns. Evening schools were common for adults who wished to learn the common branches.

School advertisements. In the *Boston Evening Post* of April 4, 1743, "Nathan Prince, Fellow of Harvard College, proposes on suitable encouragement to open a School in this Town for the instructing of young gentlemen in the most useful parts of the Mathematicks. Particularly the Elements of Geometry and Algebra; in Trigonometry and navigation, . . . " Arithmetic, surveying, and navigation were popular studies in colonial America.

Similar advertisements appear in issues of the *New York Gazette* in July and August of 1735 wherein a teacher of French agrees to teach "Reading, Writing, and Arithmetick at very reasonable Terms, which is per Quarter for Readers 5 s, for writers 8 s, for Cypherers 1 s." Occasionally a teacher advertises requesting patrons to pay the fees due, and in several instances somewhat bitter rivalry is indicated by the advertisements. In the *Philadelphia Chronicle* for 1767 one Joseph Garner, a teacher of the most useful branches of mathematics, alleges that "a malicious report has been spread" concerning his financial instability.

The following are a few of the many

RECOMMENDATIONS

of this work at large.

Dartmouth University, A. D. 1786.

AT the requeſt of Nicholas Pike, Eſq. we have inſpected his ſyſtem of Arithmetick, which we cheerfully recommend to the publick as eaſy, accurate and complete. And we apprehend there is no treatiſe of the kind extant, from which ſo great utility may ariſe to Schools.

B. WOODWARD, Math. and Phil. Prof.
JOHN SMITH, Profeſſor of the Learned Languages.

I do moſt ſincerely concur in the preceding recommendation.

J. WHEELOCK, *Preſident of the Univerſity.*

Providence, State of Rhodeiſland, 1785.

WHOEVER may have the peruſal of this treatiſe on Arithmetick may naturally conclude I might have ſpared myſelf the trouble of giving it this recommendation, as the work will ſpeak more for itſelf than the moſt elaborate recommendation from my pen can ſpeak for it; but as I have always been much delighted with the contemplation of mathematical ſubjects, and at the ſame time fully ſenſible of the utility of a work of this nature, was willing to render every aſſiſtance in my power to bring it to the publick view: And ſhould the ſtudent read it with the ſame pleaſure with which I peruſed the ſheets before they went to the Preſs, I am perſuaded he will not fail of reaping that benefit from it which he may expect, or wiſh for, to ſatisfy his curioſity in a ſubject of this nature. The Author, in treating on numbers, has done it with ſo much perſpicuity and ſingular addreſs, that I am convinced the ſtudy thereof will become more a pleaſure than a taſk.

The arrangement of the work, and the method by which he leads the *Tyro* into the firſt principles of numbers, are novelties I have not met with in any book I have ſeen. Wingate, Hatton, Ward, Hill, and many other Authors, whoſe names might be adduced, if neceſſary, have claimed a conſiderable ſhare of merit, but when brought into a comparative point of view with this treatiſe, they are inadequate and defective. This volume contains, beſides what is uſeful and neceſſary in the common affairs of life, a great fund for amuſement and entertainment. The Mechanick will find in it much more than he may have occaſion for; the Lawyer, Merchant and Mathematician, will find an ample field for the exerciſe of their genius; and I am well aſſured it may be read to great advantage by ſtudents of every claſs, from the loweſt ſchool, to the Univerſity. More than this need not be ſaid by me, and to have ſaid leſs, would be keeping back a tribute juſtly due to the merit of this work.

BENJAMIN WEST.

Univerſity in Cambridge, A. D. 1786.

HAVING, by the deſire of Nicholas Pike, Eſq. inſpected the following volume in manuſcript, we beg leave to acquaint the Publick, that in our opinion it is a work well executed, and contains a complete ſyſtem of Arithmetick. The rules are plain, and the demonſtrations perſpicuous and ſatisfactory; and we eſteem it the beſt calculated, of any ſingle piece we have met with, to lead youth, by natural and eaſy gradations, into a methodical and thorough acquaintance with the ſcience of figures. Perſons of all deſcriptions may find in it every thing, reſpecting numbers, neceſſary to their buſineſs; and not only ſo, but if they have a ſpeculative turn and mathematical taſte, may meet with much for their entertainment at a leiſure hour.

We are happy to ſee ſo uſeful an American production, which, if it ſhould meet with the encouragement it deſerves, among the inhabitants of the United States, will ſave much money in the country, which would otherwiſe be ſent to Europe, for publications of this kind.

We heartily recommend it to ſchools, and to the Community at large, and wiſh that the induſtry and ſkill of the Author may be rewarded, for ſo beneficial a work, by meeting with the general approbation and encouragement of the Publick.

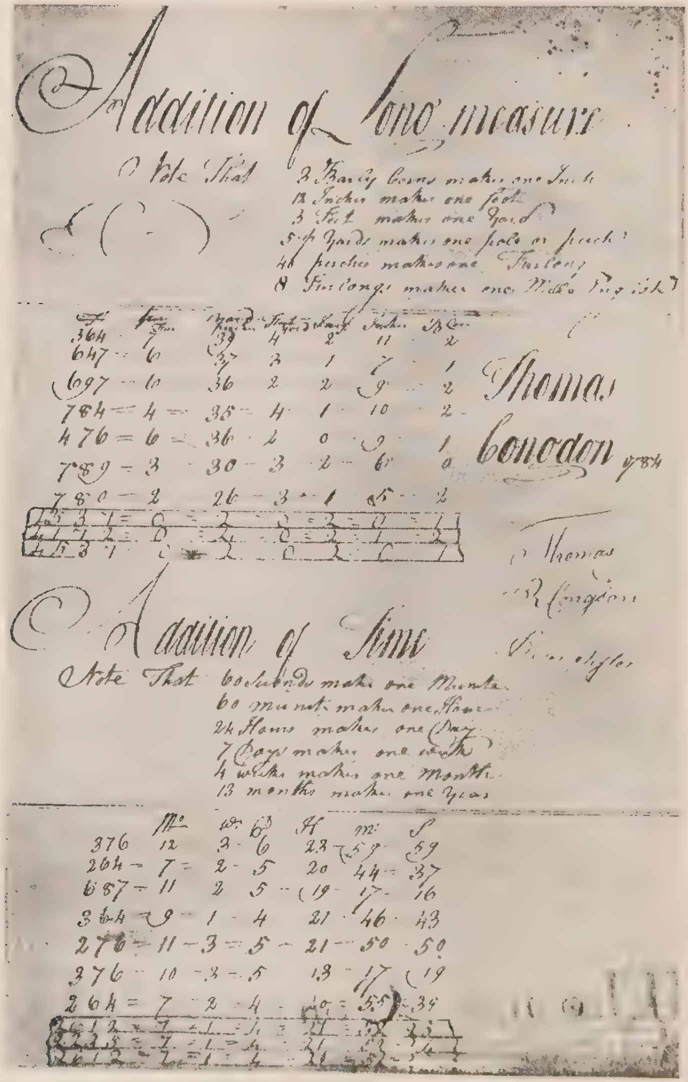
JOSEPH WILLARD, D. D. Preſident of the Univerſity.
E. WIGGLESWORTH, S. T. P. Hollis.
S. WILLIAMS, L. L. D. Math. et Phil. Nat. Prof. Hollis.

Yale College, 1786.

UPON examining Mr. Pike's Syſtem of Arithmetick and Geometry in Manuſcript, I find it to be a Work of ſuch Mathematical Ingenuity, that I eſteem myſelf honoured in joining with the Reverend Preſident Willard, and other learned Gentlemen, in recommending it to the Publick as a Production of Genius, interſperſed with Originality in this Part of Learning, and as a Book ſuitable to be taught in Schools—of Utility to the Merchant, and well adapted even for the Univerſity Inſtruction.—I conſider it of ſuch Merit, as that it will probably gain a very general Reception and Uſe throughout the Republick of Letters.

EZRA STILES, Preſident.

THE PRESIDENTS OF HARVARD, YALE, AND DARTMOUTH RECOMMEND PIKE'S ARITHMETIC



Addition of Long measure

Note That 3 Barly corns makes one Inch
12 Inches makes one foot
3 feet makes one Yard
5½ Yards makes one pole or perch
40 perches makes one Furlong
8 Furlongs makes one Mile English

364	7	39	4	2	11	2
647	6	37	3	1	7	1
697	6	36	2	2	9	2
784	4	35	4	1	10	2
476	6	36	2	0	9	1
789	3	30	3	2	6	0
780	2	26	3	1	5	2

Thomas Congdon 1784

Thomas Congdon

Addition of Time

Note That 60 seconds make one Minute
60 minutes make one Hour
24 Hours makes one Day
7 Days makes one week
4 weeks makes one Month
13 months make one year

376	12	3	6	23	57	59
264	7	2	5	20	44	37
687	11	2	5	19	17	16
364	9	1	4	21	46	43
276	11	3	5	21	50	50
376	10	3	5	13	17	19
264	7	2	4	10	55	34

EARLY NEW ENGLAND COPY BOOK OF ARITHMETIC, 1784

The *Detroit Gazette* of Friday, October 31 1823 and in other issues contains an advertisement of the projected opening on November 3rd of that year of "a Classical School at the Academy." The terms of tuition were fixed at five dollars per quarter for Greek and Latin, the same for trigonometry with mensuration, surveying and navigation, whereas "Reading, Writing, English Grammar, Geography and vulgar Arithmetick" were given at two dollars and fifty cents per quarter. The final lines of the advertisement read: "N.B. Each scholar will be required to furnish one load of wood, delivered at the Academy within one week after their admission."

In America the newspaper was, as we have seen, the medium chosen by the early teachers to reach the "patrons of science." The textbooks of the colonies did not, so far as I have been able to find, include advertisements of schools, as did many of the English texts.

Methods of instruction. Copy books were commonly employed in the schools of colonial days. At the top of a page the teacher wrote the topic under consideration; below this the pupil worked examples with a neatness not often found in the exercises of today. There is, of course, the possibility that only the finer specimens are preserved to the present day.

Arithmetic a college study. Arithmetic continued to be taught in American universities until the end of the eighteenth century, while elementary algebra continued as a college subject throughout the first half of the nineteenth century. At Yale College Jeremiah Day, who taught mathematics there in the period from 1798 to 1817, was the author of an algebra which enjoyed

long-continued popularity. Up to this time elementary arithmetic was practically a college-entrance subject.

Recognition of the teacher. The high standing of the colonial teacher of arithmetic is indicated by the distinguished supporters whose commendations are recorded in early American textbooks (*see* page 175). Though the teacher was compelled to supplement his meager earnings by outside tasks, yet in numerous ways aside from the necessary adequate financial support the community expressed its interest in the fundamentally important task of teaching arithmetic.

Bibliography for Supplementary Reading

Foster Watson, *The Beginnings of the Teaching of Modern Subjects in England*. London, Putnam and Sons, 1909.

Clifton Johnson, *Old-Time Schools and School-Books*. New York, Macmillan Co., 1904.

George E. Littlefield, *The Early Schools and Schoolbooks of New England*. Boston, Club of Odd Volumes, 1904.

W. H. Kilpatrick, "The Dutch Schools of New Netherland and Colonial New York." U. S. Bureau of Education, *Bulletin* No. 12, 1912.

E. E. Brown, *The Making of Our Middle Schools*. New York Longmans, Green and Co., 1918.

E. G. Dexter, *A History of Education in the United States*. New York, Macmillan Co., 1919.

Paul Monroe (editor), *A Cyclopedia of Education*. New York, Macmillan Co., 1919. *See* articles under Arithmetic, School, and Teacher.

THE INDEX